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# Demazure crystals of type $A_n$ and Young walls<sup>☆</sup>

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## Abstract

In this work, we give a new realization of Demazure crystals  $\mathcal{B}_w(\lambda)$  of type  $A_n$  for any Weyl group element  $w$  and for certain dominant integral weights  $\lambda$ . The vectors of  $\mathcal{B}_w(\lambda)$  are given by certain Young walls lying inside the extremal vector and which contain the highest weight vector, as diagrams. From this result, we also obtain descriptions for Demazure crystals in terms of semistandard tableaux.

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## 1. Introduction

The *Demazure module* is characterized by its highest weight  $\lambda$  and Weyl group element  $w$  pair, and is denoted by  $V_w(\lambda)$ . In [16], Littelmann conjectured a generalized character formula for Demazure modules by studying their crystal structure. Through a proof of this conjecture, Kashiwara obtained the crystal bases of the Demazure modules as subsets of crystal bases for the corresponding highest weight modules [9]. This set is called a *Demazure crystal*.

Quantum groups are deformations of the universal enveloping algebras of Kac–Moody algebras, and the crystal basis can be thought of roughly as a basis of the representation

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over quantum groups at  $q = 0$ . It is given a structure of colored oriented graph, called the crystal graph, with arrows defined by the Kashiwara operators. The crystal graphs have many nice combinatorial features reflecting the internal structure of integrable modules over quantum groups. In [10], Kashiwara and Nakashima gave an explicit realization of crystal graphs in terms of *generalized Young tableaux* for all finite dimensional irreducible modules over quantum group  $U_q(\hat{\mathfrak{g}})$ , where  $\hat{\mathfrak{g}}$  is a finite dimensional classical simple Lie algebra. Another description of crystal bases is given in [15] by Littelmann using the Lakshmibai–Seshadri monomial theory. For the quantum affine algebras, an explicit description of the crystal basis for any integrable highest weight module of type  $A_n^{(1)}$  has been given in [3,17]. This realization of crystals in terms of *paths*, which arise naturally in the context of lattice models, was generalized to other affine types in [5–7].

The characterization, in the case of  $A_1^{(1)}$ , for the Demazure crystal in terms of *extended Young diagrams* or *paths* is given in [1]. In [11–14], they study the path realization of Demazure crystals related to solvable lattice models in statistical mechanics. As an application, the characters of some families of Demazure modules are obtained for any classical affine algebras.

In [2] and [4], new combinatorial objects called the *Young walls* were introduced. Young walls may be viewed as a generalization of Young diagrams. They consist of colored blocks of various shapes built on the ground-states. The crystal graphs of basic representations for quantum affine algebra are characterized as the sets of *reduced proper Young walls*. One strong point of the Young wall description lies in the fact that the character of a representation may be computed by counting the number of colored blocks in the Young walls that have been added to the ground-state wall. In [8], a new description of crystal bases for quantum classical simple Lie algebras using the Young walls was introduced. The crystal graph for  $\hat{\mathcal{B}}(\Lambda)$  of basic representation over  $U_q(\hat{\mathfrak{g}})$  where  $\hat{\mathfrak{g}}$  is a classical affine algebra containing  $\mathfrak{g}$ , is realized as the set  $\hat{\mathcal{Y}}(\Lambda)$  of all reduced proper Young walls. It is decomposed into a disjoint union of infinitely many connected components if we take away all 0-arrows from  $\hat{\mathcal{Y}}(\Lambda)$ . Each of the connected component is isomorphic to a crystal graph for  $\mathcal{B}(\lambda)$  with the highest weight  $\lambda$  being some dominant integral weight for  $\mathfrak{g}$ . As shown in [8], every irreducible highest weight crystal for  $\mathfrak{g}$  may be obtained in this way. That is, given a dominant integral weight  $\lambda$  for  $\mathfrak{g}$ , there exists a level-1 weight  $\Lambda$  for  $\hat{\mathfrak{g}}$  such that  $\mathcal{B}(\lambda)$  appears as a connected component in  $\hat{\mathcal{Y}}(\Lambda)$  without the 0-arrows. In fact, there are many copies of  $\mathcal{B}(\lambda)$  lying inside  $\hat{\mathcal{Y}}(\Lambda)$ . The copy characterized in [8] has been chosen so that it contains Young walls made up of the least number of blocks possible.

In this work, we introduce another realization of  $\mathcal{B}(\lambda)$  in the  $A_n$  case which is different from the one given in [8]. Since the same method may be applied to show that the realizations are valid, we shall use this realization without any proof. This realization of  $\mathcal{B}(\lambda)$  has been chosen so that there exists a natural correspondence between the set of Young walls and the set of semistandard tableaux, which is a well-known realization for  $\mathcal{B}(\lambda)$ . Using this realization, we give a new realization of Demazure crystal over  $\mathfrak{sl}_{n+1}$ . In fact, we can also realize Demazure crystals using the description of crystal bases introduced in [8] following the method used in this work. First we characterize the extremal vectors of Demazure crystal for any Weyl group element and dominant integral weight. Using this characterization of extremal vectors, we also characterize the Demazure crystal  $\mathcal{B}_w(\lambda)$  of

type  $A_n$  ( $n \geq 1$ ) for any Weyl group element  $w$  and for the dominant integral weights  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$  with  $l_1, \dots, l_p$  satisfying one of the following conditions:

- $1 \leq l_1 \leq \cdots \leq l_p \leq n$  with  $p \leq 2$ ,
- $1 \leq l_1 \leq l_2 \leq l_3 = \cdots = l_p \leq n$  with  $p \geq 3$ ,
- $1 = l_1 = \cdots = l_{p'} \leq l_{p'+1} \leq l_{p'+2} = \cdots = l_p \leq n$  with  $p \geq 3$ .

The basis vectors are given by reduced Young walls lying between highest weight vector and extremal vector satisfying certain additional conditions. Specifically, if the highest weight is  $k\omega_i$  ( $1 \leq i \leq n$ ), a realization for the Demazure crystal may be written down in a very simple form. Due to our choice of realization for  $\mathcal{B}(\lambda)$ , we easily obtain one more result which is the translation of results on Demazure crystals obtained with Young walls into the language of semistandard tableaux.

The contents of this paper are organized as follows. In Section 2, we fix basic notations for quantum groups and crystal bases of type  $A_n$  and  $A_n^{(1)}$ , and review some of the basic theories of Demazure crystals of type  $A_n$ . In Section 3, we explain the combinatorics of Young walls for affine Lie algebra of types  $A_n^{(1)}$  and introduce a realization of crystal graphs for type  $A_n$  using the Young walls. Sections 4 and 5 are devoted to the statement and proof of our main results. Parts of our proof of the main theorem has been placed in the appendix.

## 2. Quantum groups and Demazure crystals

We refer the readers to the references cited in the introduction, for the basic concepts on quantum groups and crystal bases. In this section, we lay out the general setting and fix the notations used in this paper.

- $I = \{1, \dots, n\}$ : index set for simple roots of  $\mathfrak{sl}_{n+1}$ .  
 $\hat{I} = I \cup \{0\}$ : index set for simple roots of  $\hat{\mathfrak{sl}}_{n+1}$ .
- $U_q(\mathfrak{sl}_{n+1})$ : quantum group for type  $A_n$ .  
 $U_q(\hat{\mathfrak{sl}}_{n+1})$ : quantum group for type  $A_n^{(1)}$ .
- $\omega_i$ : fundamental weight for type  $A_n$ .  
 $\Lambda_i$ : fundamental weight for type  $A_n^{(1)}$ .
- $\alpha_i, \delta$ : simple root, null root.
- $P = \bigoplus_{i \in I} \mathbb{Z}\omega_i$ : weight lattice of  $\mathfrak{sl}_{n+1}$ .  
 $\hat{P} = (\bigoplus_{i \in \hat{I}} \mathbb{Z}\Lambda_i) \oplus \mathbb{Z}\delta$ : weight lattice of  $\hat{\mathfrak{sl}}_{n+1}$ .
- wt: weight for type  $A_n$ .  
 $\hat{\text{wt}}$ : (affine) weight for type  $A_n^{(1)}$ .
- $f_i, \tilde{e}_i$ : Kashiwara operators.
- $\mathcal{B}(\lambda)$ : irreducible highest weight crystal of highest weight  $\lambda$  for type  $A_n$ .  
 $\hat{\mathcal{B}}(\Lambda)$ : irreducible highest weight crystal of highest weight  $\Lambda$  for type  $A_n^{(1)}$ .

The algebra  $U_q(\mathfrak{sl}_{n+1})$  is sometimes called the *quantum classical algebra* of type  $A_n$ . Likewise, the algebra  $U_q(\hat{\mathfrak{sl}}_{n+1})$  is sometimes called the *quantum affine algebra* of type  $A_n^{(1)}$ .

Now we proceed to Demazure modules and Demazure crystals of type  $A_n$ . Let  $\{s_i\}_{i \in I}$  be the set of simple reflections corresponding to the simple roots  $\{\alpha_i\}_{i \in I}$ , and let  $W$  be the Weyl group of  $\mathfrak{sl}_{n+1}$  generated by  $\{s_i\}_{i \in I}$ . For  $w \in W$ ,  $l(w)$  denotes the length of  $w$ , and  $<$  denotes the Bruhat order on  $W$ .

Let  $U_q^+(\mathfrak{sl}_{n+1})$  be the subalgebra of  $U_q(\mathfrak{sl}_{n+1})$  generated by  $e_i$  ( $i \in I$ ). For  $\lambda \in P^+$ , we consider the irreducible highest weight  $U_q(\mathfrak{sl}_{n+1})$ -module  $V(\lambda)$ .

**Definition 2.1.** For each  $w \in W$  and  $\lambda \in P^+$ , let

$$V_w(\lambda) = U_q^+(\mathfrak{sl}_{n+1})v_{w\lambda}, \quad (2.1)$$

where  $v_{w\lambda}$  is the extremal vector of weight  $w\lambda$  of  $V(\lambda)$ . The module  $V_w(\lambda)$  is called the *Demazure module*.

For each  $w \in W$ , the  $U_q^+(\mathfrak{sl}_{n+1})$ -module  $V_w(\lambda)$  is a finite dimensional subspace of  $V(\lambda)$ .

Let  $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$  be the crystal basis of  $V(\lambda)$ . Kashiwara showed the following theorem.

**Theorem 2.2** [9]. For each  $w \in W$ , there exists a subset  $\mathcal{B}_w(\lambda)$  of  $\mathcal{B}(\lambda)$  such that

$$\frac{V_w(\lambda) \cap \mathcal{L}(\lambda)}{V_w(\lambda) \cap q\mathcal{L}(\lambda)} = \bigoplus_{b \in \mathcal{B}_w(\lambda)} \mathbf{Q}b. \quad (2.2)$$

Furthermore,  $\mathcal{B}_w(\lambda)$  has the following recursive property.

**Proposition 2.3** [9]. If  $w < s_j w$ , then

$$\mathcal{B}_{s_j w}(\lambda) = \bigcup_{m \geq 0} \tilde{f}_j^m \mathcal{B}_w(\lambda) \setminus \{0\}. \quad (2.3)$$

The set  $\mathcal{B}_w(\lambda)$  is called a *Demazure crystal*. Note that  $\dim V(\lambda)_{w\lambda} = 1$ .

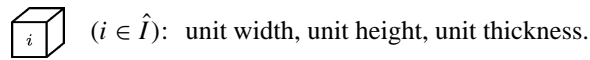
**Definition 2.4.** The unique element of  $\mathcal{B}(\lambda)$  of weight  $w\lambda$  is called the *extremal vector* of  $\mathcal{B}_w(\lambda)$ . It will be denoted by  $E_{w\lambda}$ .

### 3. Young walls

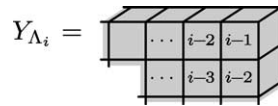
#### 3.1. Young walls for type $A_n^{(1)}$

We review the notion of Young walls for type  $A_n^{(1)}$  ( $n \geq 1$ ) and the realization of crystal graphs for basic representations over quantum affine algebras of this type using Young walls [4].

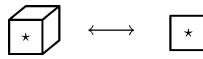
The Young walls are built of colored blocks of the following shape.



With these colored blocks, we will build a wall of thickness equal to one unit which extends infinitely to the left. Given a dominant integral weight  $\Lambda$  of level-1, we fix a wall  $Y_\Lambda$  called the *ground-state wall of weight  $\Lambda$* , extending infinitely to the left and also going infinitely downwards. For the  $\hat{\mathfrak{sl}}_{n+1}$ -case, a dominant integral weight  $\Lambda$  of level-1 is a fundamental weight  $\Lambda_i$  for some  $i \in \hat{I}$ . The ground-state wall of weight  $\Lambda_i$  is drawn below.



For convenience, we will use the following notation.



When using this notation on a wall, the blocks constituting the ground-state wall will usually not be drawn.

The rules for building the walls are given as follows.

- (1) The walls must be built on top of one of the ground-state walls (using the colored blocks).
- (2) The colored blocks should be stacked in the patterns given below for each ground-state wall.
- (3) Except for the right-most column, there should be no free space to the right of any block.

On  $Y_{\Lambda_i}$  :

					0	1
					$n$	0
					$n-1$	$n$
					$\vdots$	$\vdots$
					$\vdots$	$\vdots$
0	1	2	3	$\cdots$	$i$	$i+1$
$n$	0	1	2	$\cdots$	$i-1$	$i$

**Definition 3.1.** A wall built by adding a finite number of colored blocks to the ground-state wall  $Y_{\Lambda_i}$ , following the rules listed above, is called a *Young wall of ground-state  $\Lambda_i$* .

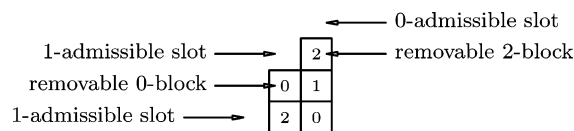
The heights of its columns will weakly decrease as we proceed from right to left. We view the blocks making up the ground-state wall to be a part of every Young wall.

Recall that  $\delta$  denoted the null root for the quantum affine algebra  $U_q(\hat{\mathfrak{sl}}_{n+1})$ . The part of a column consisting of  $(n+1)$  blocks, containing one of each of the  $(n+1)$  colors (in any cyclic order) is called a  $\delta$ -column.

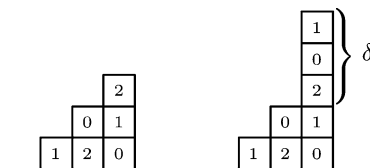
**Definition 3.2.**

- (1) A block of color  $i$  in a Young wall is called a *removable  $i$ -block* if the wall remains a Young wall after removing the block. A column in a Young wall is called  *$i$ -removable* if the top of that column is a removable  $i$ -block.
- (2) A place in a Young wall where one may add an  $i$ -block to obtain another Young wall is called an  *$i$ -admissible slot*. A column in a Young wall is called  *$i$ -admissible* if the top of that column is an  $i$ -admissible slot.
- (3) A column in a Young wall is said to *contain a removable  $\delta$*  if we may remove a  $\delta$ -column from  $Y$  and still obtain a Young wall.
- (4) A Young wall is said to be *reduced* if none of its columns contain a removable  $\delta$ .

**Example 3.3.** In the following figure, we take a Young wall of ground-state  $\Lambda_0$  for type  $A_2^{(1)}$  and indicate all the removable blocks and admissible slots.



The first Young wall drawn below is reduced, but the second one is not reduced.



For each dominant integral weight  $\Lambda$  of level-1, let  $\hat{\mathcal{Y}}(\Lambda)$  denote the set of all reduced Young walls of ground-state  $\Lambda$ .

We shall now explain the crystal structure given to  $\hat{\mathcal{Y}}(\Lambda)$ . The action of Kashiwara operators  $\tilde{e}_i, \tilde{f}_i$  ( $i \in \hat{I}$ ) on  $\hat{\mathcal{Y}}(\Lambda)$  are defined as follows. Fix  $i \in \hat{I}$  and let

$$Y = (y_k)_{k=0}^{\infty} = (\dots, y_k, \dots, y_1, y_0) \in \hat{\mathcal{Y}}(\Lambda)$$

be a reduced Young wall.

- (1) To each column  $y_k$  of  $Y$ , we assign its  $i$ -signature as follows:
  - (a) we assign  $-$  if the column is  $i$ -removable,
  - (b) we assign  $+$  if the column is  $i$ -admissible.
- (2) From the (infinite) sequence of  $+$  and  $-$ , cancel out every  $(+, -)$  pair to obtain a finite sequence of  $-$  followed by  $+$ , reading from left to right. This sequence is called the  $i$ -signature of the Young wall  $Y$ .
- (3) We define  $\tilde{e}_i Y$  to be the reduced Young wall obtained from  $Y$  by removing the  $i$ -block corresponding to the right-most  $-$  in the  $i$ -signature of  $Y$ . We define  $\tilde{e}_i Y = 0$  if there exists no  $-$  in the  $i$ -signature of  $Y$ .
- (4) We define  $\tilde{f}_i Y$  to be the reduced Young wall obtained from  $Y$  by adding an  $i$ -block to the column corresponding to the left-most  $+$  in the  $i$ -signature of  $Y$ . We define  $\tilde{f}_i Y = 0$  if there exists no  $+$  in the  $i$ -signature of  $Y$ .

Define the maps

$$\text{wt}: \hat{\mathcal{Y}}(\Lambda) \rightarrow \hat{P}, \quad \varepsilon_i: \hat{\mathcal{Y}}(\Lambda) \rightarrow \mathbf{Z}, \quad \varphi_i: \hat{\mathcal{Y}}(\Lambda) \rightarrow \mathbf{Z} \quad (3.1)$$

on the set  $\hat{\mathcal{Y}}(\Lambda)$  by

$$\text{wt}(Y) = \Lambda - \sum_{i \in \hat{I}} k_i \alpha_i, \quad (3.2)$$

$$\varepsilon_i(Y) = \text{the number of } - \text{ in the } i\text{-signature of } Y, \quad (3.3)$$

$$\varphi_i(Y) = \text{the number of } + \text{ in the } i\text{-signature of } Y, \quad (3.4)$$

where  $k_i$  is the number of  $i$ -blocks in  $Y$  that have been added to the ground-state wall  $Y_\Lambda$ . Then we obtain:

**Theorem 3.4** [3,4,17]. *Let  $\Lambda$  be a level-1 dominant integral weight for  $\hat{\mathfrak{sl}}_{n+1}$ .*

- (1) *The maps*

$$\text{wt}: \hat{\mathcal{Y}}(\Lambda) \rightarrow \hat{P}, \quad \tilde{e}_i, \tilde{f}_i: \hat{\mathcal{Y}}(\Lambda) \rightarrow \hat{\mathcal{Y}}(\Lambda) \cup \{0\}, \quad \varepsilon_i, \varphi_i: \hat{\mathcal{Y}}(\Lambda) \rightarrow \mathbf{Z}$$

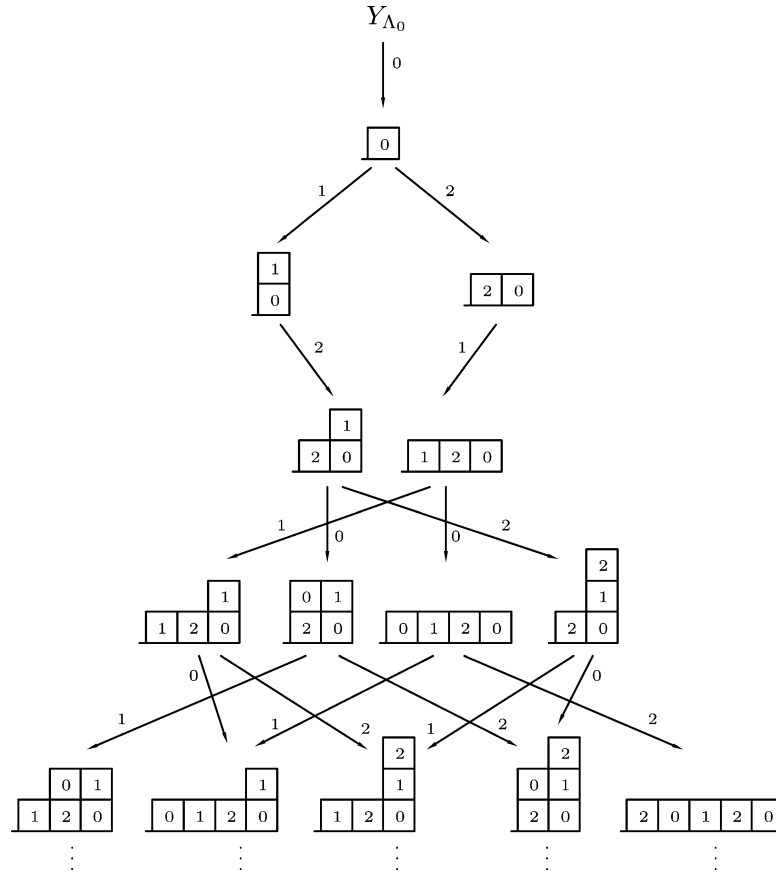
*define a  $U_q(\hat{\mathfrak{sl}}_{n+1})$ -crystal structure on the set  $\hat{\mathcal{Y}}(\Lambda)$ .*

- (2) *There exists an isomorphism of  $U_q(\hat{\mathfrak{sl}}_{n+1})$ -crystals*

$$\hat{\mathcal{Y}}(\Lambda) \xrightarrow{\sim} \hat{\mathcal{B}}(\Lambda)$$

*sending  $Y_\Lambda \mapsto u_\Lambda$ , where  $u_\Lambda$  is the highest weight vector in  $\hat{\mathcal{B}}(\Lambda)$ .*

**Example 3.5.** Following is the top part of graph of  $U_q(\hat{\mathfrak{sl}}_3)$ -crystal  $\hat{\mathcal{Y}}(\Lambda_0)$ .



### 3.2. Young walls for type $A_n$

In [8] a realization of crystal graphs for irreducible highest weight representations over quantum classical algebra  $U_q(\mathfrak{sl}_{n+1})$  ( $n \geq 1$ ) was given using Young walls. In this work, we introduce another realization different from the one given in [8]. Since the same methods may be applied to show that the realizations are valid, we shall state it without any proof.

For each  $i \in \hat{I}$ , the crystal graph for  $\hat{\mathcal{B}}(\Lambda_i)$  is realized as the set  $\hat{\mathcal{Y}}(\Lambda_i)$  of all reduced Young walls by Theorem 3.4. It is decomposed into a disjoint union of infinitely many connected components if we take away all 0-arrows from  $\hat{\mathcal{Y}}(\Lambda_i)$ . Each of the connected component is isomorphic to a crystal graph for  $\mathcal{B}(\lambda)$  with the highest weight  $\lambda$  being some dominant integral weight for  $\mathfrak{sl}_{n+1}$ .

As we shall see, every irreducible highest weight crystal for  $\mathfrak{sl}_{n+1}$  may be obtained in this way. That is, given a dominant integral weight  $\lambda$  for  $\mathfrak{sl}_{n+1}$ , there exists a weight  $\Lambda_i$  for  $\hat{\mathfrak{sl}}_{n+1}$  such that  $\mathcal{B}(\lambda)$  appears as a connected component in  $\hat{\mathcal{Y}}(\Lambda_i)$  without the 0-arrows.



In fact, there are many copies of  $\mathcal{B}(\lambda)$  lying inside  $\hat{\mathcal{Y}}(\Lambda_i)$ . The copy characterized in this work has been chosen so that there exists a natural correspondence between the set of Young walls and the set of semistandard tableaux, which is a well-known realization for  $\mathcal{B}(\lambda)$ .

Let us define a partial order on  $\hat{\mathcal{Y}}(\Lambda_i)$  with  $i \in \hat{I}$ . Given  $Y \in \hat{\mathcal{Y}}(\Lambda_i)$ , we shall let  $y_k$  denote the  $k$ th column of  $Y$ , counting from right to left. For  $Y = (y_k)_{k=0}^\infty$  and  $Y' = (y'_k)_{k=0}^\infty$  belonging to  $\hat{\mathcal{Y}}(\Lambda_i)$ , we will write  $Y \preceq Y'$  if and only if each column  $y_k$  is not taller than  $y'_k$ . In more plain terms, we write  $Y \preceq Y'$  if and only if the diagram  $Y$  is contained in the diagram  $Y'$ .

We explain this concept with an example.

**Example 3.6.** Consider the Young walls

$$Y = (y_k)_{k=0}^\infty, \quad Y' = (y'_k)_{k=0}^\infty, \quad Y'' = (y''_k)_{k=0}^\infty, \quad (3.5)$$

for  $A_2^{(1)}$  belonging to  $\hat{\mathcal{Y}}(\Lambda_0)$ , where

$$Y = \begin{array}{|c|c|c|} \hline 1 & 2 & 0 \\ \hline \end{array}, \quad Y' = \begin{array}{|c|c|c|c|} \hline & & 0 & 1 \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array}, \quad Y'' = \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array}.$$

Since each column  $y_k$  is not taller than  $y'_k$  (or  $y''_k$ ), we have  $Y \preceq Y'$  (respectively  $Y \preceq Y''$ ). On the other hand, since the column  $y''_0$  is strictly taller than  $y'_0$  and  $y'_1$  is strictly taller than  $y''_1$ , there is no order relation between  $Y'$  and  $Y''$ .

Given a dominant integral weight  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p} \in P^+$  for  $\mathfrak{sl}_{n+1}$  with  $1 \leq l_1 \leq \cdots \leq l_p \leq n$ , we describe an algorithm for constructing the vector  $H_\lambda$  and vector  $L_\lambda$  belonging to the  $U_q(\hat{\mathfrak{sl}}_{n+1})$ -crystal  $\hat{\mathcal{Y}}(\Lambda_r)$ , where  $r \equiv l_1 + \cdots + l_p \pmod{n+1}$ .

First, in the case  $\lambda = \omega_i$  ( $i = 1, \dots, n$ ), let  $H_{\omega_i}$  be the ground-state wall  $Y_{\Lambda_i}$ , and let  $L_{\omega_i}$  be the following Young wall.

$$\begin{array}{|c|c|c|} \hline n-i+1 & \cdots & n \\ \hline & & \\ \hline & & \\ \hline 1 & \cdots & i \\ \hline \end{array}$$

This Young wall is the largest Young wall that may be built on top of the ground-state wall  $Y_{\Lambda_i}$  without using any 0-blocks. You can see that the 0th and  $i$ th columns contain 0-slots.

Now, suppose  $\lambda$  has the form  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$  ( $1 \leq l_1 \leq \cdots \leq l_p \leq n$ ). The Young wall  $H_\lambda$  (and  $L_\lambda$ ) is obtained in the following way. First break off the first  $l_2$ -many columns from the wall  $H_{\omega_{l_2}}$  (respectively  $L_{\omega_{l_2}}$ ). Then, shift the detached columns upward by  $(n - l_1 + 1)$  units and attach it to the right of  $H_{\omega_{l_1}}$  (respectively  $L_{\omega_{l_1}}$ ). Next, break off the first  $l_3$ -many columns from the wall  $H_{\omega_{l_3}}$  (respectively  $L_{\omega_{l_3}}$ ), shift the detached columns upward by  $(2n - (l_1 + l_2) + 2)$  units, and attach it to the right of the previously constructed wall. Continue in a similar way until we have attached shifted  $l_p$ -many columns, broken off from  $H_{\omega_{l_p}}$  (respectively  $L_{\omega_{l_p}}$ ), to the previous construction. The resulting wall is set to  $H_\lambda$  (respectively  $L_\lambda$ ).

From now on, when drawing a Young wall containing  $H_\lambda$ , we shall give an outline of  $H_\lambda$  with a dark thick line.

**Example 3.7.** Let  $\lambda = 2\omega_1 + \omega_2$  be a weight for  $\mathfrak{sl}_3$ . Then the vectors  $H_\lambda$  and  $L_\lambda$  are elements of the  $U_q(\hat{\mathfrak{sl}}_3)$ -crystal  $\hat{\mathcal{Y}}(\Lambda_r)$ , and given below. Here  $r \equiv 1 + 1 + 2 \pmod{3}$ .

$$H_\lambda = \begin{array}{|c|c|c|} \hline & 0 & 1 \\ \hline & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \quad L_\lambda = \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline & 2 & \\ \hline & 1 & \\ \hline 2 & & \\ \hline 1 & & \\ \hline \end{array} \quad \begin{array}{c} \leftarrow H_\lambda \end{array}$$

Notice that the columns constituting the reduced Young wall  $L_\lambda$  may be naturally grouped with the height of the columns in mind. There are  $p$  such groups of columns if we exclude the group of columns which has the same height as the ground-state wall. For an element  $Y \in \hat{\mathcal{Y}}(\Lambda_r)$  such that  $H_\lambda \preceq Y \preceq L_\lambda$ , we will denote by  $Y^j$  the group of blocks from  $Y$  that lie inside the  $j$ th group of columns mentioned above, counting from left to right.

The  $j$ th component  $Y^j$  has  $l_j$  columns. Its  $k$ th column, counting from left to right, will be denoted by  $Y_k^j$ . Given a column  $Y_k^j$ , we denote by  $c(Y_k^j)$  the color of the top block in the column.

**Example 3.8.** Let  $\lambda$  be given as in Example 3.7. The shaded Young wall  $Y$  given below is a vector of the  $U_q(\hat{\mathfrak{sl}}_3)$ -crystal  $\hat{\mathcal{Y}}(\Lambda_1)$  satisfying,  $H_\lambda \preceq Y \preceq L_\lambda$ .

$$Y = \begin{array}{|c|c|c|c|} \hline & 1 & 2 & \\ \hline & 2 & 0 & 1 \\ \hline & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & 2 \\ \hline 1 & 2 & 0 & 1 \\ \hline \end{array} \quad \begin{array}{c} \leftarrow L_\lambda \\ \leftarrow H_\lambda \end{array}$$

For the Young wall  $Y$ , the components  $Y^j$  are given as follows.

$$Y^1 = \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \quad Y^2 = \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 2 \\ \hline \end{array} \quad Y^3 = \begin{array}{|c|c|} \hline & 2 \\ \hline 0 & 1 \\ \hline 2 & 0 \\ \hline 1 & 2 \\ \hline 0 & 1 \\ \hline \end{array}$$

We have omitted the ground-state wall parts in the above figures of  $Y^j$ .

Finally, we let  $\mathcal{Y}(\lambda)$  be the set of elements  $Y \in \hat{\mathcal{Y}}(\Lambda_r)$  satisfying the conditions:

- (F1)  $H_\lambda \preceq Y \preceq L_\lambda$ ,
- (F2)  $c(Y_k^{j+1}) \leq c(Y_k^j)$  for all  $j \in \{1, \dots, p-1\}$  and  $k \in \{1, \dots, l_j\}$ .

In particular,  $\mathcal{Y}(\omega_i)$  is the set of all reduced Young walls lying inside  $L_{\omega_i}$  and which contain  $H_{\omega_i}$ .

Let  $Z_1, \dots, Z_p$  be parts from some set of reduced Young walls for  $A_n^{(1)}$ . Let  $Z_j$  contain  $l_j$  columns and suppose that they satisfy the following conditions:

- $c((H_\lambda)_k^j) \leq c((Z_j)_k) \leq c((L_\lambda)_k^j)$  for each  $j \in \{1, \dots, p\}$  and  $k \in \{1, \dots, l_j\}$ ,
- $c((Z_{j+1})_k) \leq c((Z_j)_k)$  for each  $j \in \{1, \dots, p-1\}$  and  $k \in \{1, \dots, l_j\}$ .

Here, we wrote the  $k$ th column of  $Z_j$ , counting from left to right, as  $(Z_j)_k$ . Now, it is clear that there exists a unique element  $Y \in \mathcal{Y}(\lambda)$  such that  $Y^j = Z_j$  for each  $j$ . We shall denote such a  $Y$  by  $[Z_1, \dots, Z_p]$ . For an element  $Y \in \mathcal{Y}(\lambda)$ , we can clearly write  $Y = [Y^1, \dots, Y^p]$ .

The crystal graph for  $\mathcal{B}(\lambda)$  of the irreducible highest weight representation is realized as the following set of reduced Young walls.

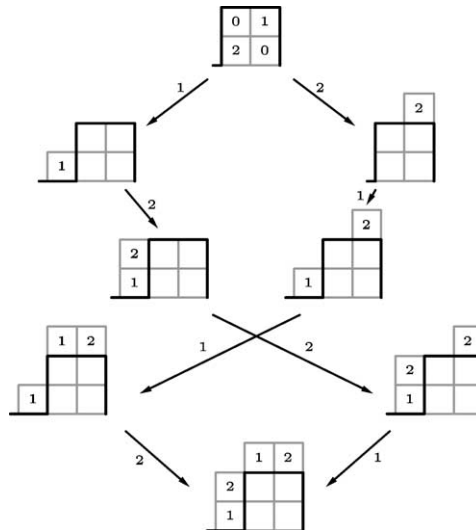
**Theorem 3.9** (cf. [8]). *Let  $\lambda \in P^+$ . Then there is an isomorphism of  $U_q(\mathfrak{sl}_{n+1})$ -crystals*

$$\mathcal{Y}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda), \quad (3.6)$$

sending  $H_\lambda \mapsto u_\lambda$ , where  $u_\lambda$  is the highest weight vector of  $\mathcal{B}(\lambda)$ .

From now on, we will call  $H_\lambda$  and  $L_\lambda$ , the highest weight vector and lowest weight vector of the crystal  $\mathcal{Y}(\lambda)$ , respectively.

**Example 3.10.** The following is the graph of  $U_q(\mathfrak{sl}_3)$ -crystal  $\mathcal{Y}(\omega_1 + \omega_2)$ . The graph appears as one connected component in the  $U_q(\mathfrak{sl}_3)$ -crystal  $\hat{\mathcal{Y}}(\Lambda_0)$  when we remove the 0-arrows.



Starting from the fifth line of figure in Example 3.5, we can find a copy of this graph.

#### 4. Demazure crystals

In this section, we realize the Demazure crystal of type  $A_n$ . Through Theorem 3.9, we know  $\mathcal{B}(\lambda)$  and  $\mathcal{Y}(\lambda)$  are isomorphic as crystals. Let us denote by  $\mathcal{Y}_w(\lambda)$ , the set of elements from  $\mathcal{Y}(\lambda)$  which corresponds to elements of  $\mathcal{B}_w(\lambda) \subset \mathcal{B}(\lambda)$ . We shall now describe the set  $\mathcal{Y}_w(\lambda)$  for  $\mathfrak{sl}_{n+1}$ . In fact, we can also realize Demazure crystal using description of crystal bases introduced in [8] following the method used in this work.

As in  $\mathcal{B}_w(\lambda)$ , we shall denote by  $E_{w\lambda}$ , the extremal vector of  $\mathcal{Y}_w(\lambda)$ .

##### 4.1. Extremal vectors

First, we realize the extremal vectors for any  $w \in W$  and  $\lambda \in P^+$  for  $\mathfrak{sl}_{n+1}$ . Throughout this subsection, we fix a weight  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$  with  $1 \leq l_1 \leq \cdots \leq l_p \leq n$ .

For the  $U_q(\mathfrak{sl}_{n+1})$ -crystal  $\mathcal{Y}(\omega_i)$ , if  $Y \in \mathcal{Y}(\omega_i)$  has a  $j$ -admissible slot for  $j \in I$ , then  $\varphi_j(Y) = 1$  and  $\tilde{f}_j^{\varphi_j(Y)} Y$  will be the reduced Young wall obtained from  $Y$  by adding a  $j$ -block. Notice that,  $\varphi_j(Y) = 0$  or  $1$  for any  $Y \in \mathcal{Y}(\omega_i)$ . That is,  $Y$  has at most one  $j$ -admissible slot.

First, we state the following well-known basic fact.

**Lemma 4.1.** *Let  $w = s_{i_r} s_{i_{r-1}} \cdots s_{i_1} \in W$  be written in reduced form. Consider the following recursive formula*

$$Y_1 = \tilde{f}_{i_1}^{\varphi_{i_1}(H_{\omega_i})} H_{\omega_i}, \quad (4.1)$$

$$Y_k = \tilde{f}_{i_k}^{\varphi_{i_k}(Y_{k-1})} Y_{k-1} \quad \text{for } k = 2, \dots, r. \quad (4.2)$$

The extremal vector  $E_{w\omega_i}$  of  $\mathcal{Y}_w(\omega_i)$  is given by the element  $Y_r$  in  $\mathcal{Y}(\omega_i)$ .

**Proof.** We shall use induction on  $w$  to show  $\text{wt}(Y_r) = w\omega_i$ . It is clear that  $\text{wt}(Y_1) = \omega_i - \varphi_{i_1}(H_{\omega_i})\alpha_{i_1} = s_{i_1}\omega_i$ . Assume  $\text{wt}(Y_{r-1}) = w'\omega_i$  where  $w' = s_{i_{r-1}} \cdots s_{i_1}$ . We then have

$$\begin{aligned} \text{wt}(Y_r) &= \text{wt}(Y_{r-1}) - \varphi_{i_r}(Y_{r-1})\alpha_{i_r} = \text{wt}(Y_{r-1}) - \text{wt}(Y_{r-1})(h_{i_r})\alpha_{i_r} \\ &= w'\omega_i - w'\omega_i(h_{i_r})\alpha_{i_r} = s_{i_r}w'\omega_i = w\omega_i. \end{aligned}$$

The second equality uses the fact that  $w'$  was written in a reduced expression. This completes the induction step.  $\square$

In terms of notation introduced in the previous section,  $E_{w\omega_i}$  is given as follows

$$E_{w\omega_i} = [(E_{w\omega_i})^1]. \quad (4.3)$$

The first (and unique) component  $(E_{w\omega_i})^1$  has  $i$  columns and its columns will be written as  $(E_{w\omega_i})_k^1$ .

The following proposition is convenient for determining the extremal vectors.

**Proposition 4.2.** For  $w \in W$ ,

$$E_{w\lambda} = [(E_{w\omega_{l_1}})^1, \dots, (E_{w\omega_{l_p}})^1], \quad (4.4)$$

that is,  $(E_{w\lambda})^j = (E_{w\omega_{l_j}})^1$  for each  $j \in \{1, \dots, p\}$ .

**Proof.** For each  $j$  and  $k$ ,  $c((H_\lambda)_k^j) \leq c((E_{w\omega_{l_j}})_k^1) \leq c((L_\lambda)_k^j)$ . And by use of Lemma 4.1, we may obtain

$$c((E_{w\omega_{l_{j+1}}})_k^1) \leq c((E_{w\omega_{l_j}})_k^1)$$

for each  $j \in \{1, \dots, p-1\}$  and  $k \in \{1, \dots, l_j\}$ . Thus we know  $[(E_{w\omega_{l_1}})^1, \dots, (E_{w\omega_{l_p}})^1]$  is a well-defined vector of  $\mathcal{Y}(\lambda)$ . On the other hand, let  $b_1^j, \dots, b_{u(j)}^j$  be colors of the blocks which is contained in the first (and unique) component  $(E_{w\omega_{l_j}})^1$  of  $E_{w\omega_{l_j}}$ , and which is not contained in the ground-state wall. Then we have

$$\begin{aligned} \text{wt}([(E_{w\omega_{l_1}})^1, \dots, (E_{w\omega_{l_p}})^1]) \\ &= \lambda - (\alpha_{b_1^1} + \dots + \alpha_{b_{u(1)}^1}) - (\alpha_{b_1^2} + \dots + \alpha_{b_{u(2)}^2}) - \dots - (\alpha_{b_1^p} + \dots + \alpha_{b_{u(p)}^p}) \\ &= (\omega_{l_1} - (\alpha_{b_1^1} + \dots + \alpha_{b_{u(1)}^1})) + \dots + (\omega_{l_p} - (\alpha_{b_1^p} + \dots + \alpha_{b_{u(p)}^p})) \\ &= w\omega_{l_1} + \dots + w\omega_{l_p} = w\lambda. \end{aligned}$$

Hence we have  $[(E_{w\omega_{l_1}})^1, \dots, (E_{w\omega_{l_p}})^1] = E_{w\lambda}$ .  $\square$

**Remark 4.3.** From Lemma 4.1 and Proposition 4.2, we obtain the following facts. Let  $w \prec s_i w$ .

- (1) Extremal vector  $E_{s_i w \lambda}$  has no  $i$ -admissible slot.
- (2) The corresponding components of the extremal vectors  $E_{w\lambda}$  and  $E_{s_i w \lambda}$  are different by at most one  $i$ -block. In the case that they are different, the  $i$ -block is a removable block of  $E_{s_i w \lambda}$ .

**Example 4.4.** The shaded Young walls given below are the extremal vectors for the  $U_q(\mathfrak{sl}_6)$ -Demazure crystals  $\mathcal{Y}_w(\omega_2)$ ,  $\mathcal{Y}_w(\omega_3)$ , and  $\mathcal{Y}_w(\omega_4)$  corresponding to  $w = s_2 s_3 s_2 s_5 s_4 s_3$ . They easily may be obtained from Lemma 4.1.

$$E_{w\omega_2} = \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 3 & 4 \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \quad E_{w\omega_3} = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 2 & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \quad E_{w\omega_4} = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

And from Proposition 4.2, we can easily obtain the extremal vector  $E_{w(\omega_2+\omega_3+\omega_4)} = [(E_{w\omega_2})^1, (E_{w\omega_3})^1, (E_{w\omega_4})^1]$  of crystal  $\mathcal{Y}_w(\omega_2 + \omega_3 + \omega_4)$ .

$$E_{w(\omega_2+\omega_3+\omega_4)} = \begin{array}{ccccccccc} & & & & & 2 & 3 & 4 & 5 \\ & & & & & 1 & 2 & 3 & 4 \\ & & 3 & 4 & 5 & 0 & & & 3 \\ & 2 & 3 & 4 & & & 0 & & \\ & 1 & 2 & 3 & & & & 0 & \\ 4 & 5 & 0 & & & 3 & & & 0 \\ 3 & 4 & & 0 & & & 3 & & \\ 2 & 3 & & & 0 & & & 3 & \\ 1 & 2 & 3 & & & 0 & & & 3 \end{array}$$

For any  $i \in I$ , set  $I_{(i)} = \{1, \dots, i\}$ . For a  $Y \in \mathcal{Y}(\lambda)$ , given a component  $Y^k$  of  $Y$ , notice that there is at most one  $i$ -admissible slot and at most one removable  $i$ -block. We use the following notations:

$$Y^k \nearrow \boxed{i} : \text{component obtained by placing an } i\text{-block in } Y^k, \quad (4.5)$$

$$Y^k \nwarrow \boxed{i} : \text{component obtained by removing an } i\text{-block from } Y^k. \quad (4.6)$$

We shall only use these notations when the result obtained by adding or removing an  $i$ -block is a valid component of Young walls obtained by acting  $\tilde{f}_i$  or  $\tilde{e}_i$  (many times, if necessary) on  $Y$ .

The following proposition shows that we may find all extremal vectors in  $\mathcal{Y}(\lambda)$ .

**Proposition 4.5.**

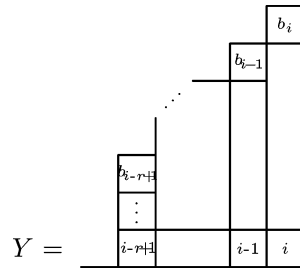
- (1) Fix  $i \in I$ . Any element of  $\mathcal{Y}(\omega_i)$  is an extremal vector  $E_{w\omega_i}$  for some  $w \in W$ .
- (2) A Young wall  $Y \in \mathcal{Y}(\lambda)$  is an extremal vector  $E_{w\lambda}$  for some  $w \in W$  if and only if  $Y$  is an element of  $\mathcal{Y}(\lambda)$  satisfying the following condition: there exists some family of strictly increasing maps

$$\phi_{a,a+1} : I_{(l_a)} \rightarrow I_{(l_{a+1})}$$

$$\text{satisfying } c(Y_k^a) = c(Y_{\phi_{a,a+1}(k)}^{a+1}) \text{ for all } a \in \{1, \dots, p-1\} \text{ and } k \in I_{(l_a)}.$$

**Proof.** (1) Let  $Y$  be any element of  $\mathcal{Y}(\omega_i)$ . Considering just the blocks that have been added on top of the ground-state wall, let  $Y$  be made up of  $r$  columns. Here,  $0 \leq r \leq i$

because of the condition (F1). In the case  $r = 0$ ,  $Y = H_{\omega_i} = E_{\omega_i}$ . For other cases, let  $b_k = c(Y_k^1)$  for each  $k \in \{i - r + 1, \dots, i - 1, i\}$ .



Now, set

$$w = s_{b_{i-r+1}} \cdots s_{i-r+2} s_{i-r+1} \cdots s_{b_{i-1}} \cdots s_i s_{i-1} s_{b_i} \cdots s_{i+1} s_i. \quad (4.7)$$

This expression of  $w$  is reduced. By Lemma 4.1, we have  $Y = E_{w\omega_i}$  for this  $w \in W$ .

(2) We first show the if part. Fix any  $Y$  satisfying the condition. We construct a sequence of colors in the following manner.

(Step 0) Set  $k = l_p$ .

(Step 1) Among the columns of  $Y$ , single out the columns, whose color of the top block is  $c(Y_k^p)$ .

(Step 2) Take the left-most column among the columns taken and read from bottom to top, the colors of the blocks in the column, not contained in the highest weight vector.

(Step 3) Go back to (Step 1) with  $k - 1$  in place of  $k$  if  $k \neq 1$ , and stop if  $k = 1$ .

Since  $Y$  satisfies the given condition, all columns of  $Y$  has been dealt with in (Step 1) of this algorithm. Also, if a column belonging to some component was chosen in (Step 2), in every component that comes to the right of this component, there always exist a column whose top block is of the same color as of the top block of the chosen column.

Let us denote the sequence of colors by  $\{b_j\}_{j=1}^l$  obtained from this algorithm. Set  $w = s_{b_l} \cdots s_{b_1}$ . This expression of  $w$  is reduced. By Proposition 4.2, we obtain  $Y = E_{w\lambda}$ . To help the readers in understanding the above argument, we have illustrated it in Example 4.7.

Next we show the remaining part using induction on  $w$ . Let  $r$  be the number of  $\omega_i$  in  $\lambda$ . And if  $r \neq 0$ , we let  $q$  be the smallest number  $k$  with  $l_k = i$ .

In the case  $r = 0$ ,  $E_{s_i\lambda} = H_\lambda$ . We define  $\phi_{a,a+1}$  for all  $a \in \{1, \dots, p-1\}$  by setting

$$\phi_{a,a+1}(x) = x \quad \text{when } x \in I_{(l_a)}. \quad (4.8)$$

Otherwise, by Proposition 4.2, we have

$$E_{s_i\lambda} = \left[ (H_\lambda)^1, \dots, (H_\lambda)^{q-1}, (H_\lambda)^q \swarrow \boxed{i}, \dots, (H_\lambda)^{q+r-1} \swarrow \boxed{i}, (H_\lambda)^{q+r}, \dots, (H_\lambda)^p \right].$$

For the extremal vector  $E_{s_i\lambda}$ , we define the map  $\phi_{a,a+1}$  for each  $a \in \{1, \dots, p-1\}$  by setting

$$\phi_{q+r-1,q+r}(x) = \begin{cases} x & \text{when } x \in I_{(l_{q+r-1})} \setminus \{l_{q+r-1}\}, \\ l_{q+r-1} + 1 & \text{when } x = l_{q+r-1}, \end{cases} \quad (4.9)$$

and for the remaining  $a$

$$\phi_{a,a+1}(x) = x \quad \text{when } x \in I_{(l_a)}. \quad (4.10)$$

Then,  $E_{s_i\lambda}$  satisfies the condition with the family of maps  $\{\phi_{a,a+1}\}$ .

Now, assume  $E_{w\lambda}$  satisfies the condition with a family of maps  $\{\psi_{a,a+1}\}$  and let  $w \prec s_i w$ . We construct the new maps  $\{\tau_{a,a+1}\}$  as follows. In the case  $E_{w\lambda} = E_{s_i w\lambda}$ , we set

$$\tau_{a,a+1} = \psi_{a,a+1} \quad \text{for all } a \in \{1, \dots, p-1\}. \quad (4.11)$$

Otherwise, we know that  $E_{s_i w\lambda}$  contains removable  $i$ -blocks which is not contained in  $E_{w\lambda}$  by Remark 4.3. Let the columns of  $E_{s_i w\lambda}$  which contain the removable  $i$ -blocks be  $(E_{s_i w\lambda})_{u_1}^{j_1}, \dots, (E_{s_i w\lambda})_{u_l}^{j_l}$ . Here  $j_1, \dots, j_l$  are all distinct. For each  $m \in \{1, \dots, l\}$ , unless  $j_m = p$ , we define  $\tau_{j_m, j_m+1}$  as follows. If  $j_m + 1 = j_k$  for some  $k \in \{1, \dots, l\}$ , then  $c((E_{s_i w\lambda})_{u_m}^{j_m}) = c((E_{s_i w\lambda})_{\psi_{j_m, j_m+1}(u_m)}^{j_m+1})$ . For such  $m$ , we set

$$\tau_{j_m, j_m+1} = \psi_{j_m, j_m+1}. \quad (4.12)$$

In the case,  $j_m + 1 \neq j_k$  for any  $k \in \{1, \dots, l\}$ , we have

$$c((E_{s_i w\lambda})_{\psi_{j_m, j_m+1}(u_m)}^{j_m+1}) < c((E_{s_i w\lambda})_{u_m}^{j_m}) \quad \text{and} \quad c((E_{s_i w\lambda})_{\psi_{j_m, j_m+1}(u_m)+1}^{j_m+1}) = i$$

by Remark 4.3. For such  $m$ , we set

$$\tau_{j_m, j_m+1}(x) = \begin{cases} \psi_{j_m, j_m+1}(x) & \text{when } x \in \{1, \dots, l_{j_m}\} \setminus \{u_m\}, \\ \psi_{j_m, j_m+1}(x) + 1 & \text{when } x = u_m. \end{cases} \quad (4.13)$$

And for each  $a \in \{1, \dots, p-1\} \setminus \{j_1, \dots, j_l\}$ , set

$$\tau_{a,a+1} = \psi_{a,a+1}. \quad (4.14)$$

Then we can easily check that  $E_{s_i w\lambda}$  satisfies the condition with the family of maps  $\{\tau_{a,a+1}\}$ . The induction on  $w$  is complete.  $\square$

**Remark 4.6.** The family of maps  $\{\phi_{a,a+1}\}$  in Proposition 4.5(2) is unique. Suppose another family of maps  $\{\psi_{a,a+1}\}$  satisfies the condition. Then there exists some  $a$  such that  $\phi_{a,a+1} \neq \psi_{a,a+1}$ . This means either  $(E_{w\lambda})^a$  or  $(E_{w\lambda})^{a+1}$  has at least two shapes. This is contradiction. Hence, each extremal vector has only one family of maps  $\{\phi_{a,a+1}\}$  satisfying the condition in Proposition 4.5(2).



**Example 4.7.** Following shaded Young wall  $Y$  is an element of  $U_q(\mathfrak{sl}_5)$ -crystal  $\mathcal{Y}(\omega_1 + \omega_2 + \omega_3)$ . The condition of Proposition 4.5(2) is satisfied with the strictly increasing maps

$$\phi_{1,2}: I_{(1)} \rightarrow I_{(2)} \quad \text{given by} \quad \phi_{1,2}(1) = 2, \quad (4.15)$$

$$\phi_{2,3}: I_{(2)} \rightarrow I_{(3)} \quad \text{given by} \quad \phi_{2,3}(1) = 2, \phi_{2,3}(2) = 3. \quad (4.16)$$

$$Y = \begin{array}{ccccccc} & & & & 2 & 3 & 4 \\ & & & & 1 & 2 & 3 \\ & 3 & 4 & 0 & & 2 & \\ & 2 & 3 & & 0 & & \\ & 1 & 2 & & & & 0 \\ 4 & 0 & & 2 & & & \\ 3 & & 0 & & 2 & & \\ 2 & & & 0 & & 2 & \\ 1 & 2 & & & 0 & & \end{array}$$

By applying the algorithm given in the proof of Proposition 4.5 to  $Y$ , we obtain the sequence  $(1, 2, 3, 4, 1, 2, 1)$ . So we have  $w = s_1 s_2 s_1 s_4 s_3 s_2 s_1$ . And using Proposition 4.2, we can confirm  $Y = E_{w(\omega_1 + \omega_2 + \omega_3)}$ .

**Corollary 4.8.**

- (1) The number of distinct extremal vectors in  $\mathcal{Y}(\omega_i)$  is  $\binom{n+1}{i}$ .
- (2) The number of distinct extremal vectors in  $\mathcal{Y}(\lambda)$  is given by  $\binom{n+1}{l_p} \binom{l_p}{l_{p-1}} \binom{l_{p-1}}{l_{p-2}} \cdots \binom{l_2}{l_1}$ .

**Proof.** These statements are immediate from Proposition 4.5.  $\square$

The following theorem characterizes the extremal vectors.

**Theorem 4.9.** Fix  $w \in W$ . Let  $\mathcal{G}$  be the set of elements  $Y \in \mathcal{Y}(\lambda)$  satisfying the following two conditions:

- (1)  $Y$  has a family of strictly increasing maps

$$\phi_{a,a+1}: I_{(l_a)} \rightarrow I_{(l_{a+1})}$$

satisfying  $c(Y_k^a) = c(Y_{\phi_{a,a+1}(k)}^{a+1})$  for all  $a \in \{1, \dots, p-1\}$  and  $k \in I_{(l_a)}$ ,

- (2) the sequence  $\{b_r\}_{r=1}^l$  obtained from  $Y$  by applying the algorithm given in the proof of Proposition 4.5(2) satisfies  $s_{b_l} \cdots s_{b_1} < w$  or  $s_{b_l} \cdots s_{b_1} = w$ .

Then  $E_{w\lambda}$  is the (unique) element  $Y'$  of  $\mathcal{G}$  satisfying  $Y \preccurlyeq Y'$  for all  $Y \in \mathcal{G}$ .

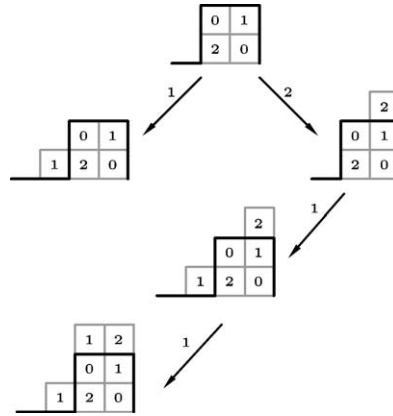
**Proof.** We can easily check this statement using Propositions 4.2 and 4.5(2).  $\square$

The Weyl group element  $s_{b_l} \cdots s_{b_1}$  obtained from  $E_{w\lambda}$  by applying the algorithm given in the proof of Proposition 4.5(2), does not always equal  $w$ . For example, from the extremal vector  $E_{w'(\omega_2+\omega_3+\omega_4)}$  with  $w' = s_2s_3s_2s_4s_5s_4s_3$ , we obtain the sequence  $(3, 4, 5, 2, 3, 2)$ , and  $s_2s_3s_2s_5s_4s_3 < w'$ .

#### 4.2. Realization

In this section, we realize the Demazure crystal  $\mathcal{Y}_w(\lambda)$  for certain  $\lambda \in P^+$ .

**Example 4.10.** Following is the graph of  $\mathcal{Y}_{s_1s_2}(\omega_1 + \omega_2)$  of type  $A_2$  obtained from the definition of Demazure crystals.



Final Young wall in the above figure is the extremal vector  $E_{s_1s_2(\omega_1+\omega_2)}$ .

The following theorem characterizes the Demazure crystals for some family of  $\lambda \in P^+$ .

**Theorem 4.11.** Fix any  $w \in W$ . Let  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$  with  $l_1, \dots, l_p$  satisfying one of the following conditions.

- (1)  $1 \leq l_1 \leq \cdots \leq l_p \leq n$  with  $p \leq 2$ .
- (2)  $1 \leq l_1 \leq l_2 \leq l_3 = \cdots = l_p \leq n$  with  $p \geq 3$ .
- (3)  $1 = l_1 = \cdots = l_{p'} \leq l_{p'+1} \leq l_{p'+2} = \cdots = l_p \leq n$  with  $p \geq 3$ .

For each  $\lambda$ , the crystal  $\mathcal{Y}_w(\lambda)$  is the set of elements  $Y \in \mathcal{Y}(\lambda)$  satisfying the following conditions.

(Y1)  $Y \preceq E_{w\lambda}$ .

(Y2) There exists some family of strictly increasing maps  $\{\phi_{a,b} : I_{(l_a)} \rightarrow I_{(l_b)} \mid 1 \leq a < b \leq p\}$  such that

(Y2-1)  $c(Y_{\phi_{a,b}(u)}^b) \leq c(Y_u^a) \leq c((E_{w\lambda})_{\phi_{a,b}(u)}^b)$  for each  $u \in I_{(l_a)}$ ,

(Y2-2) for  $1 \leq a < b < c \leq p$ , if  $\phi_{a,c}(u) = \phi_{b,c}(v)$  for some  $u \in I_{(l_a)}$  and  $v \in I_{(l_b)}$ , then  $c(Y_v^b) \leq c(Y_u^a)$ .

The proof of this theorem is given in the next section. We may conclude from the above theorem that any two Demazure crystals with the same extremal vectors must be the same.

**Example 4.12.** Following shaded Young wall is an element of  $U_q(\mathfrak{sl}_6)$ -crystal  $\mathcal{Y}(\omega_2 + \omega_3 + \omega_4)$ .

$$Y =$$

						1	2	3	4
						0	1	2	3
				3	4	5	0	1	2
		1	2	3	4	5	0	1	
		0	1	2	3	4	5	0	
	4	5	0	1	2	3	4	5	
	3	4	5	0	1	2	3	4	
	1	2	3	4	5	0	1	2	3

For  $w = s_1 s_2 s_3 s_2 s_4 s_3 s_2$ , the Young wall is contained in the extremal vector of weight  $w(\omega_2 + \omega_3 + \omega_4)$ , which is marked by the light, thick line. And the Young wall  $Y$  satisfies condition (Y2) of the Theorem 4.11 with strictly increasing maps

$$\phi_{1,2}: I_{(2)} \rightarrow I_{(3)} \quad \text{given by} \quad \phi_{1,2}(1) = 1, \phi_{1,2}(2) = 2, \quad (4.17)$$

$$\phi_{1,3}: I_{(2)} \rightarrow I_{(4)} \quad \text{given by} \quad \phi_{1,3}(1) = 1, \phi_{1,3}(2) = 2, \quad (4.18)$$

$$\phi_{2,3}: I_{(3)} \rightarrow I_{(4)} \quad \text{given by} \quad \phi_{2,3}(1) = 1, \phi_{2,3}(2) = 2, \phi_{2,3}(3) = 3. \quad (4.19)$$

Thus by the above theorem, the Young wall  $Y$  is an element of  $\mathcal{Y}_w(\omega_2 + \omega_3 + \omega_4)$ .

Notice that no two of the conditions given in Theorem 4.11 imply the remaining condition. We will confirm this using some examples.

**Remark 4.13.** The Young walls  $Y_1$ ,  $Y_2$ , and  $Y_3$  given below are vectors of the  $U_q(\mathfrak{sl}_5)$ -crystal  $\mathcal{Y}(2\omega_2 + \omega_3)$ .

For  $w = s_2 s_3 s_2$ , the following shaded Young wall  $Y_1$  satisfies condition (Y2), but does not satisfy (Y1). And  $Y_1$  is not an element of  $\mathcal{Y}_{s_2 s_3 s_2}(2\omega_2 + \omega_3)$ .

$$Y_1 =$$

								4
							2	3
					0	1	2	
			3	4	0	1		
			2	3	4	0		
		0	1	2	3	4		
	3	4	0	1	2	3		
	2	3	4	0	1	2		

$\leftarrow E_{s_2 s_3 s_2}(2\omega_2 + \omega_3)$

Following shaded Young wall  $Y_2$  satisfies all conditions except (Y2-1) of the above theorem. All family of strictly increasing maps  $\{\phi_{a,b}\}$  satisfying condition (Y2-2), do not

satisfy the condition (Y2-1). Actually, there does not exist any family of strictly increasing maps  $\{\phi_{a,b}\}$  satisfying condition (Y2-1). Young wall  $Y_2$  is not an element of  $\mathcal{Y}_w(2\omega_2 + \omega_3)$  for  $w = s_1 s_2 s_4 s_3 s_2$ .

$E_{w(2\omega_2 + \omega_3)}$  ←

$$Y_2 = \begin{array}{cccccccc} & & & & & & & 4 \\ & & & & & & 1 & 2 & 3 \\ & & & & & 4 & 0 & 1 & 2 \\ & & & & 3 & 4 & 0 & 1 \\ & & 1 & 2 & 3 & 4 & 0 \\ & 4 & 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 & 1 & 2 \end{array}$$

The following shaded Young wall  $Y_3$  is contained in  $E_{w(2\omega_2 + \omega_3)}$ . And there exists a unique family of strictly increasing maps  $\{\phi_{a,b}\}$  satisfying the condition (Y2-1). But the family  $\{\phi_{a,b}\}$  satisfying condition (Y2-1), does not satisfy condition (Y2-2). The Young wall  $Y_3$  is not an element of  $\mathcal{Y}_w(2\omega_2 + \omega_3)$  where  $w = s_3 s_2 s_1 s_4 s_3 s_2$ .

$E_{w(2\omega_2 + \omega_3)}$  ←

$$Y_3 = \begin{array}{cccccccc} & & & & & & & 3 & 4 \\ & & & & & & 2 & 3 \\ & & & & 3 & 4 & 0 & 1 & 2 \\ & & & 2 & 3 & 4 & 0 & 1 \\ & & 1 & 2 & 3 & 4 & 0 \\ & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 & 1 & 2 \end{array}$$

In particular, for the following  $\lambda$ , we can write down Theorem 4.11 in a simple form.

**Remark 4.14.** Fix any  $w \in W$ .

- (1)  $\mathcal{Y}_w(\omega_i) = \{Y \in \mathcal{Y}(\omega_i) \mid Y \preceq E_{w\omega_i}\}$ .
- (2)  $\mathcal{Y}_w(\omega_i + \omega_j)$  with  $i \leq j$  is the set of element  $Y \in \mathcal{Y}(\omega_i + \omega_j)$  satisfying the following conditions:
  - (a)  $Y \preceq E_{w(\omega_i + \omega_j)}$ ,
  - (b) there exists a strictly increasing map  $\phi: I_{(i)} \rightarrow I_{(j)}$  such that  $c(Y_{\phi(k)}^2) \leq c(Y_k^1) \leq c((E_{w(\omega_i + \omega_j)})_{\phi(k)}^2)$  for all  $k \in I_{(i)}$ .

We introduce a number of lemmas which is needed to prove our main theorem.

**Lemma 4.15.** Fix any  $\lambda \in P^+$  and  $w \in W$ . Suppose  $w \prec s_i w$ . If  $Y$  and  $\tilde{f}_i Y$  are elements of  $\mathcal{Y}(\lambda)$  with  $Y \preceq E_{s_i w \lambda}$ , then  $\tilde{f}_i Y \preceq E_{s_i w \lambda}$ .

**Proof.** Suppose not. Since the nonzero element  $\tilde{f}_i Y$  is obtained from  $Y$  by adding an  $i$ -block and  $Y \preceq E_{s_i w \lambda}$ , the  $i$ -block sits in an  $i$ -admissible slot of  $E_{s_i w \lambda}$ . But we know  $E_{s_i w \lambda}$  has no  $i$ -admissible slot, from Remark 4.3. So this is a contradiction.  $\square$

**Lemma 4.16.**

- (1) Fix  $w \in W$  and  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$  with  $1 \leq l_1 \leq l_2 \leq l_3 = \cdots = l_p \leq n$  and  $p \geq 3$ . Let  $Y \in \mathcal{Y}(\lambda)$  with  $Y \preceq E_{w\lambda}$ . Then  $Y$  satisfies the condition (Y2) if and only if  $Y$  satisfies following condition:
- (A2) there exists some family of strictly increasing maps  $\{\phi_{a,b} : I_{(l_a)} \rightarrow I_{(l_b)} \mid 1 \leq a < b \leq 3\}$  such that
- (A2-1)  $c(Y_{\phi_{a,b}(u)}^b) \leq c(Y_u^a) \leq c((E_{w\lambda})_{\phi_{a,b}(u)}^b)$  for  $u \in I_{(l_a)}$ ,
- (A2-2) for  $1 \leq a < b < c \leq 3$ , if  $\phi_{a,c}(u) = \phi_{b,c}(v)$  for some  $u \in I_{(l_a)}$  and  $v \in I_{(l_b)}$ , then  $c(Y_v^b) \leq c(Y_u^a)$ .
- (2) Fix  $w \in W$  and  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$  with  $1 = l_1 = \cdots = l_{p'} \leq l_{p'+1} \leq l_{p'+2} = \cdots = l_p \leq n$  and  $p \geq 3$ . Let  $Y \in \mathcal{Y}(\lambda)$  with  $Y \preceq E_{w\lambda}$ . Then  $Y$  satisfies the condition (Y2) if and only if  $Y$  satisfies following condition:
- (B2) there exists some family of strictly increasing maps  $\{\phi_{a,b} : I_{(l_a)} \rightarrow I_{(l_b)} \mid 1 \leq a \leq p' + 1, p' + 1 \leq b \leq p' + 2, \text{ and } a < b\}$  such that
- (B2-1)  $c(Y_{\phi_{a,b}(u)}^b) \leq c(Y_u^a) \leq c((E_{w\lambda})_{\phi_{a,b}(u)}^b)$  for  $u \in I_{(l_a)}$ ,
- (B2-2) for  $1 \leq a \leq p', p' + 1 \leq b < c \leq p' + 2$ , if  $\phi_{a,c}(u) = \phi_{b,c}(v)$  for some  $u \in I_{(l_a)}$  and  $v \in I_{(l_b)}$ , then  $c(Y_v^b) \leq c(Y_u^a)$ .

**Proof.** (1) The only if part is trivial. Let  $Y \in \mathcal{Y}(\lambda)$  satisfies the condition (A2) with  $\{\phi_{a,b} \mid 1 \leq a < b \leq 3\}$ . We show that  $Y$  has a family of strictly increasing maps  $\{\psi_{a,b} \mid 1 \leq a < b \leq p\}$  satisfying conditions (Y2-1) and (Y2-2).

We define the map  $\psi_{a,b} : I_{(l_a)} \rightarrow I_{(l_b)}$  for each pair  $(a, b)$  with  $1 \leq a < b \leq p$ , by setting

$$\psi_{a,b} = \phi_{a,b} \quad (4.20)$$

where  $1 \leq a < b \leq 3$ ,

$$\psi_{a,b} = \phi_{a,3} \quad (4.21)$$

where  $1 \leq a \leq 2$  and  $4 \leq b \leq p$ , and

$$\psi_{a,b}(x) = x \quad \text{when } x \in I_{(l_a)} \quad (4.22)$$

where  $3 \leq a < b \leq p$ . Then we can show that  $\psi_{a,b}$  satisfies the conditions (Y2-1) and (Y2-2) for  $Y$ , by using Proposition 4.2 and the facts  $Y \in \mathcal{Y}(\lambda)$ ,  $Y \preceq E_{w\lambda}$ , and  $Y$  satisfies (A2).

(2) The only if part is trivial. Let  $Y \in \mathcal{Y}(\lambda)$  satisfies condition (B2) with  $\{\phi_{a,b} \mid 1 \leq a \leq p' + 1, p' + 1 \leq b \leq p' + 2, \text{ and } a < b\}$ . We show that  $Y$  has a family of strictly increasing maps  $\{\psi_{a,b} \mid 1 \leq a < b \leq p\}$  satisfying conditions (Y2-1) and (Y2-2).

We define the map  $\psi_{a,b} : I_{(l_a)} \rightarrow I_{(l_b)}$  for each pair  $(a, b)$  with  $1 \leq a < b \leq p$ , by setting

$$\psi_{a,b} = \phi_{a,b} \quad (4.23)$$

where  $1 \leq a \leq p' + 1$ ,  $p' + 1 \leq b \leq p' + 2$ , and  $a < b$ ,

$$\psi_{a,b}(x) = x \quad \text{when } x \in I_{(l_a)} \quad (4.24)$$

where  $1 \leq a < b \leq p'$ ,

$$\psi_{a,b} = \phi_{a,p'+2} \quad (4.25)$$

where  $1 \leq a \leq p' + 1$  and  $p' + 3 \leq b \leq p$ , and

$$\psi_{a,b}(x) = x \quad \text{when } x \in I_{(l_a)} \quad (4.26)$$

where  $p' + 2 \leq a < b \leq p$ . Then we can show that  $\psi_{a,b}$  satisfies the conditions (Y2-1) and (Y2-2) for  $Y$ , by using Proposition 4.2 and the facts  $Y \in \mathcal{Y}(\lambda)$ ,  $Y \preceq E_{w\lambda}$ , and  $Y$  satisfies (B2).  $\square$

If the highest weight is  $k\omega_i$  ( $1 \leq i \leq n$ ), a realization for the Demazure crystal may be written down in a very simple form.

**Corollary 4.17.** *For any  $w \in W$  and  $i \in I$ ,*

$$\mathcal{Y}_w(k\omega_i) = \{Y \in \mathcal{Y}(k\omega_i) \mid Y \preceq E_{w(k\omega_i)}\}. \quad (4.27)$$

That is,  $\mathcal{Y}_w(k\omega_i)$  is the set of all reduced Young walls in  $\mathcal{Y}(k\omega_i)$  lying between  $H_{k\omega_i}$  and  $E_{w(k\omega_i)}$ .

#### 4.3. Semistandard tableaux

In this section, we realize the Demazure crystal  $\mathcal{B}_w(\lambda)$  over  $\mathfrak{sl}_{n+1}$  for certain  $\lambda \in P^+$  using semistandard tableaux.

For any  $\lambda \in P^+$ , let  $\mathcal{S}(\lambda)$  be the set consisting of all semistandard tableaux of shape  $\lambda$ . The set  $\mathcal{S}(\lambda)$  is a well known realization for the crystal basis  $\mathcal{B}(\lambda)$  over  $\mathfrak{sl}_{n+1}$  [10].

To represent an element  $T \in \mathcal{S}(\lambda)$  with  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$ , we will write

$$T = (T_{i,j}), \quad (4.28)$$

where  $T_{i,j}$  stands for the  $(i, j)$ -entry of  $T$ , as in a matrix.

In the following proposition, we give a crystal isomorphism between two kinds of descriptions of the crystal basis  $\mathcal{B}(\lambda)$ , the set of Young walls  $\mathcal{Y}(\lambda)$  and the set of semistandard tableaux  $\mathcal{S}(\lambda)$ .

**Proposition 4.18** [18]. Let  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p} \in P^+$  with  $1 \leq l_1 \leq \cdots \leq l_p \leq n$ . Then there exists an isomorphism of  $U_q(\mathfrak{sl}_{n+1})$ -crystals

$$\mathcal{Y}(\lambda) \xrightarrow{\sim} \mathcal{S}(\lambda) \quad (4.29)$$

given by  $Y \mapsto T$ , where  $T_{k,p-a+1} = c(Y_k^a) + 1$  for each  $a \in \{1, \dots, p\}$  and  $k \in I_{(l_a)}$ .

**Example 4.19.** We give an example for the (one to one) correspondence given in Proposition 4.18. The following is a semistandard tableau corresponding to the Young wall  $E_{w(\omega_2+\omega_3+\omega_4)}$  given in Example 4.4.

1	1	1
3	4	4
4	6	
6		

Let us denote by  $\mathcal{S}_w(\lambda)$ , the set of elements from  $\mathcal{S}(\lambda)$  which corresponds to elements of  $\mathcal{B}_w(\lambda) \subset \mathcal{B}(\lambda)$ . As in  $\mathcal{B}_w(\lambda)$  and  $\mathcal{Y}_w(\lambda)$ , we shall also denote by  $E_{w\lambda}$ , the extremal vector of  $\mathcal{S}_w(\lambda)$ .

Now, we can translate all facts about Demazure crystals presented using Young walls that were introduced in the previous section into the language of semistandard tableaux. In particular, the two corollaries given below are obtained from Theorems 4.9 and 4.11 of previous sections.

Fix a  $T \in \mathcal{S}(\lambda)$  which has a family of strictly increasing maps

$$\phi_{a,a+1}: I_{(l_a)} \rightarrow I_{(l_{a+1})} \quad (4.30)$$

satisfying  $T_{k,p-a+1} = T_{\phi_{a,a+1}(k),p-a}$  for all  $a \in \{1, \dots, p-1\}$  and  $k \in I_{(l_a)}$ . Here  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$  with  $1 \leq l_1 \leq \cdots \leq l_p \leq n$ . The following is the analogue of the algorithm introduced in the proof of Proposition 4.5(2). We construct a sequence of blocks in the following manner.

(Step 0) Set  $k = l_p$ .

(Step 1) Single out all blocks from  $T$ , whose color is  $T_{k,1}$ .

(Step 2) Let the position of the right most block chosen be  $(i, j)$ . And let  $H_\lambda$  be the highest weight vector of  $\mathcal{S}(\lambda)$ . Notice that  $(H_\lambda)_{i,j} \leq T_{i,j}$  and  $T_{i,j} = T_{k,1}$ . Write down the colors  $(H_\lambda)_{i,j}, (H_\lambda)_{i,j} + 1, \dots, T_{i,j} - 1$ .

(Step 3) Go back to (Step 1) with  $k-1$  in place of  $k$  if  $k \neq 1$ , and stop if  $k = 1$ .

Let us denote the sequence of colors obtained from this algorithm by  $\{b_r\}_{r=1}^t$ .

**Corollary 4.20.** Fix  $w \in W$  and  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p} \in P^+$  with  $1 \leq l_1 \leq \cdots \leq l_p \leq n$ . Let  $\mathcal{G}'$  be the set of elements  $T \in \mathcal{S}(\lambda)$  satisfying the following two conditions.

- $T$  has a family of strictly increasing maps

$$\phi_{a,a+1} : I_{(l_a)} \rightarrow I_{(l_{a+1})}$$

satisfying  $T_{k,p-a+1} = T_{\phi_{a,a+1}(k),p-a}$  for all  $a \in \{1, \dots, p-1\}$  and  $k \in I_{(l_a)}$ .

- The sequence  $\{b_r\}_{r=1}^l$  obtained from  $T$  by applying the algorithm given above satisfies  $s_{b_l} \cdots s_{b_1} \prec w$  or  $s_{b_l} \cdots s_{b_1} = w$ .

Then  $E_{w\lambda}$  is the (unique) element  $T'$  of  $\mathcal{G}'$  satisfying  $T_{i,j} \leq T'_{i,j}$  for any  $i, j$  and  $T \in \mathcal{G}'$ .

The following is a characterization of the Demazure crystal in terms of semistandard tableaux.

**Corollary 4.21.** Fix any  $w \in W$ . Let  $\lambda = \omega_{l_1} + \cdots + \omega_{l_p}$  with  $l_1, \dots, l_p$  satisfying one of the following conditions.

- (1)  $1 \leq l_1 \leq \cdots \leq l_p \leq n$  with  $p \leq 2$ .
- (2)  $1 \leq l_1 \leq l_2 \leq l_3 = \cdots = l_p \leq n$  with  $p \geq 3$ .
- (3)  $1 = l_1 = \cdots = l_{p'} \leq l_{p'+1} \leq l_{p'+2} = \cdots = l_p \leq n$ , with  $p \geq 3$ .

For each  $\lambda$ , the crystal  $\mathcal{S}_w(\lambda)$  is the set of elements  $T \in \mathcal{S}(\lambda)$  satisfying the following conditions.

- (S1)  $T_{i,j} \leq (E_{w\lambda})_{i,j}$  for all  $i, j$ .
- (S2) There exists some family of strictly increasing maps  $\{\phi_{a,b} : I_{(l_a)} \rightarrow I_{(l_b)} \mid 1 \leq a < b \leq p\}$  such that
  - (S2-1)  $T_{\phi_{a,b}(u),p-b+1} \leq T_{u,p-a+1} \leq (E_{w\lambda})_{\phi_{a,b}(u),p-b+1}$  for each  $u \in I_{(l_a)}$ ,
  - (S2-2) for  $1 \leq a < b < c \leq p$ , if  $\phi_{a,c}(u) = \phi_{b,c}(v)$  for some  $u \in I_{(l_a)}$  and  $v \in I_{(l_b)}$ , then  $T_{v,p-b+1} \leq T_{u,p-a+1}$ .

**Remark 4.22.** For any  $w \in W$ ,  $i \in I$ , and  $l \in \mathbf{Z}_{>0}$ , the Demazure crystal  $\mathcal{S}_w(l\omega_i)$  is the set of all semistandard tableau  $T$  of shape  $l\omega_i$  satisfying  $T_{k,l} \leq (E_{w\omega_i})_{k,1}$  for all  $k$ .

## 5. Proof of theorem

This section is devoted to proving Theorem 4.11.

Recall that we have three types of  $\lambda$  to deal with. The proof for  $\lambda$  of type (1) is similar to and simpler than the proof for  $\lambda$  of types (2) and (3). So we shall only deal with  $\lambda$  of types (2) and (3). In the course of the proof for this theorem, where possible, we shall deal with the cases (2) and (3) simultaneously.

Let us write  $\mathcal{A}_w(\lambda)$  for the set of elements of  $\mathcal{Y}(\lambda)$  satisfying two given conditions (Y1) and (Y2) given in Theorem 4.11. For each  $Y \in \mathcal{Y}(\lambda)$  satisfying (Y2), we let  $\mathcal{M}(Y, w)$  be the set of family of maps  $\{\phi_{a,b}\}$ , satisfying two given conditions (Y2-1) and (Y2-2). Similarly, for each  $Y \in \mathcal{Y}(\lambda)$  satisfying the condition (A2) (respectively (B2)), given in Lemma 4.16,



we let  $\mathcal{M}_A(Y, w)$  (respectively  $\mathcal{M}_B(Y, w)$ ) be the set of family of maps  $\{\phi_{a,b}\}$ , satisfying two given conditions (A2-1), (A2-2) (respectively (B2-1), (B2-2)).

We use induction on  $w$ . First we will show that  $\mathcal{Y}_{s_i}(\lambda) = \mathcal{A}_{s_i}(\lambda)$ . Let  $r$  be the number of  $\omega_i$  in  $\lambda$  and if there exists a number  $k$  with  $l_k = i$ , we let  $q$  be the smallest such  $k$ . Then we have

$$\mathcal{Y}_{s_i}(\lambda) = \{Y_0, Y_1, \dots, Y_r\} \quad (5.1)$$

$$= \{Y \in \mathcal{Y}(\lambda) \mid Y \preceq Y_r\}, \quad (5.2)$$

where

$$Y_0 = [(H_\lambda)^1, \dots, (H_\lambda)^p],$$

$$Y_j = \left[ (H_\lambda)^1, \dots, (H_\lambda)^{q-1}, (H_\lambda)^q \swarrow \boxed{i}, \dots, (H_\lambda)^{q+j-1} \swarrow \boxed{i}, (H_\lambda)^{q+j}, \dots, (H_\lambda)^p \right]$$

for each  $j \in \{1, \dots, r\}$ . Notice that,  $Y_0$  is  $H_\lambda$  and  $E_{s_i \lambda}$  is equal to  $Y_r$ . Now since  $\mathcal{Y}_{s_i}(\lambda)$  has all elements  $Y$  of  $\mathcal{Y}(\lambda)$  satisfying condition (Y1), we have  $\mathcal{A}_{s_i}(\lambda) \subset \mathcal{Y}_{s_i}(\lambda)$ .

Now, let us show the converse inclusion. For each element  $Y_j$  of  $\mathcal{Y}_{s_i}(\lambda)$ , we define the maps  $\{\phi_{a,b} : I_{(l_a)} \rightarrow I_{(l_b)} \mid 1 \leq a < b \leq p\}$ . Set

$$\phi_{a,b}(x) = \begin{cases} x & \text{when } x \in I_{(l_a)} \setminus \{l_a\}, \\ l_a + 1 & \text{when } x = l_a, \end{cases} \quad (5.3)$$

where  $q \leq a \leq q + j - 1$  and  $b > q + r - 1$ . For the remaining pairs  $(a, b)$ , we set

$$\phi_{a,b}(x) = x \quad \text{when } x \in I_{(l_a)}. \quad (5.4)$$

Then the defined family  $\{\phi_{a,b}\} \in \mathcal{M}(Y_j, s_i)$ , for each  $Y_j$  of  $\mathcal{Y}_{s_i}(\lambda)$ . This shows  $\mathcal{Y}_{s_i}(\lambda) \subset \mathcal{A}_{s_i}(\lambda)$  and therefore  $\mathcal{Y}_{s_i}(\lambda) = \mathcal{A}_{s_i}(\lambda)$ .

We now proceed with the induction step. Assume  $\mathcal{Y}_w(\lambda) = \mathcal{A}_w(\lambda)$  and  $w < s_i w$ . By induction hypothesis,

$$\mathcal{Y}_{s_i w}(\lambda) = \bigcup_{m \geq 0} \tilde{f}_i^m \mathcal{Y}_w(\lambda) \setminus \{0\} \quad (5.5)$$

$$= \bigcup_{m \geq 0} \tilde{f}_i^m \mathcal{A}_w(\lambda) \setminus \{0\}. \quad (5.6)$$

It suffices to show

$$\mathcal{A}_{s_i w}(\lambda) = \bigcup_{m \geq 0} \tilde{f}_i^m \mathcal{A}_w(\lambda) \setminus \{0\}. \quad (5.7)$$

We shall show Eq. (5.7) in two steps.

**Step 1.** We shall first show that the right-hand side in (5.7) is contained in  $\mathcal{A}_{s_i w}(\lambda)$ . Here, we use induction on  $m$ .

For each  $Y \in \mathcal{A}_w(\lambda)$  and any family of maps  $\{\phi_{a,b}\} \in \mathcal{M}(Y, w)$ , we have

$$Y \preceq E_{w\lambda} \preceq E_{s_i w\lambda} \quad (5.8)$$

and

$$c(Y_{\phi_{a,b}(u)}^b) \leq c(Y_u^a) \leq c((E_{w\lambda})_{\phi_{a,b}(u)}^b) \leq c((E_{s_i w\lambda})_{\phi_{a,b}(u)}^b) \quad (5.9)$$

for any pair  $(a, b)$  and  $u \in I_{(l_a)}$ . So we have  $\{\phi_{a,b}\} \in \mathcal{M}(Y, s_i w)$ . This shows  $Y \in \mathcal{A}_{s_i w}(\lambda)$  and therefore  $\mathcal{A}_w(\lambda) \subset \mathcal{A}_{s_i w}(\lambda)$ .

Assume that

$$\tilde{f}_i^m \mathcal{A}_w(\lambda) \setminus \{0\} \subset \mathcal{A}_{s_i w}(\lambda). \quad (5.10)$$

Given an element  $Y' \in \tilde{f}_i^{m+1} \mathcal{A}_w(\lambda) \setminus \{0\}$ , we have  $Y' = \tilde{f}_i Y$  for some  $Y \in \tilde{f}_i^m \mathcal{A}_w(\lambda) \setminus \{0\}$ . Then by induction hypothesis,  $Y \in \mathcal{A}_{s_i w}(\lambda)$ . From Lemma 4.15, we obtain  $\tilde{f}_i Y \preceq E_{s_i w\lambda}$ , so  $\tilde{f}_i Y$  satisfies the condition (Y1) for  $s_i w$ . We should now show that  $\tilde{f}_i Y$  satisfies the condition (Y2) for  $s_i w$ . This will be done for each type of  $\lambda$ , separately, using Lemma 4.16.

Recall that we can write  $Y = [Y^1, \dots, Y^p]$ . Let

$$\tilde{f}_i Y = [Y^1, \dots, Y^k \swarrow \boxed{i}, \dots, Y^p]. \quad (5.11)$$

We assume that the  $u$ th column of the  $k$ th component of  $\tilde{f}_i Y$  contains the  $i$ -block added to  $Y$ .

$$(\tilde{f}_i Y)^k = Y^k \swarrow \boxed{i} = \begin{array}{c} \text{Diagram showing a Young diagram with a box labeled } i \text{ added to the } k\text{th component.} \end{array} \quad (\tilde{f}_i Y)_u^k$$

**For  $\lambda$  of type (2).** By Lemma 4.16, it is enough to show that  $\tilde{f}_i Y$  satisfies condition (A2) for  $s_i w$ . Since  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , there exists  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ . Fix a  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ . If the family of maps  $\{\phi_{a,b}\} \in \mathcal{M}_A(\tilde{f}_i Y, s_i w)$ , then  $\tilde{f}_i Y$  satisfies the condition (A2) for  $s_i w$ . So we obtain  $\tilde{f}_i Y \in \mathcal{A}_{s_i w}(\lambda)$ . If, to the contrary, the family of maps  $\{\phi_{a,b}\} \notin \mathcal{M}_A(\tilde{f}_i Y, s_i w)$ , then  $\{\phi_{a,b}\}$  does not satisfy at least one of the conditions (A2-1) or (A2-2) for  $\tilde{f}_i Y$ .

**Case 1.** First, we consider the case when  $\{\phi_{a,b}\}$  satisfies condition (A2-1) but does not satisfy the condition (A2-2) for  $\tilde{f}_i Y$ .

It means  $k = 2$  and for some  $v \in I_{(l_1)}$

$$\phi_{1,3}(v) = \phi_{k,3}(u), \quad c((\tilde{f}_i Y)_u^k) > c((\tilde{f}_i Y)_v^1). \quad (5.12)$$

For this case,  $c((\tilde{f}_i Y)_v^1) = c(Y_v^1) = i - 1$ . We also know that  $v + 1 \in I_{(l_1)}$  and  $c((\tilde{f}_i Y)_{v+1}^1) = c(Y_{v+1}^1) = i$  since  $\tilde{f}_i$  acts on  $Y^k$ . Set

$$\tau_{1,3}(x) = \begin{cases} \phi_{1,3}(v) + 1 & \text{when } x = v + 1, \\ \phi_{1,3}(x) & \text{when } x \in I_{(l_1)} \setminus \{v + 1\}. \end{cases} \quad (5.13)$$

For  $x \in I_{(l_k)}$ , set

$$\tau_{k,3}(x) = \begin{cases} \phi_{k,3}(x) + 1 & \text{when } \phi_{1,3}(v) \leq \phi_{k,3}(x) \leq \phi_{1,3}(v + 1), \\ \phi_{k,3}(x) & \text{otherwise.} \end{cases} \quad (5.14)$$

And finally set

$$\tau_{1,k} = \phi_{1,k}. \quad (5.15)$$

Note that

$$\{\phi_{k,3}(x) \mid \phi_{1,3}(v) \leq \phi_{k,3}(x) \leq \phi_{1,3}(v + 1), x \in I_{(l_k)}\} \cap \{\phi_{1,3}(v + 1)\} = \emptyset.$$

We can show  $\{\tau_{a,b}\} \in \mathcal{M}_A(\tilde{f}_i Y, s_i w)$  by using the fact  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$  so that  $\tilde{f}_i Y$  satisfies the condition (A2) for  $s_i w$ .

**Case 2.** Next we deal with the case when  $\{\phi_{a,b}\}$  does not satisfy condition (A2-1) for  $\tilde{f}_i Y$ . It means that we have the following three cases.

One case is when  $k = 1$  and for some  $b$  with  $k < b \leq 3$ ,

$$c((\tilde{f}_i Y)_u^k) > c((E_{s_i w \lambda})_{\phi_{k,b}(u)}^b), \quad (5.16)$$

second case is when  $k = 2$  and for some  $b$  with  $k < b \leq 3$ ,

$$c((\tilde{f}_i Y)_u^k) > c((E_{s_i w \lambda})_{\phi_{k,b}(u)}^b), \quad (5.17)$$

and the remaining case is when  $k = 3$  and for some  $a$  with  $1 \leq a < k$ ,

$$c((\tilde{f}_i Y)_u^k) > c((\tilde{f}_i Y)_v^a) \quad (5.18)$$

where  $v \in I_{(l_a)}$  satisfies  $\phi_{a,k}(v) = u$ .

For the case  $k = 1$ , we have  $c((E_{s_i w \lambda})_{\phi_{k,b}(u)}^b) = i - 1$ . And also we obtain  $\phi_{k,b}(u) + 1 \in I_{(l_b)}$  and  $c((E_{s_i w \lambda})_{\phi_{k,b}(u)+1}^b) = i$  from Remark 4.3. Set

$$\tau_{k,b}(x) = \begin{cases} \phi_{k,b}(u) + 1 & \text{when } x = u, \\ \phi_{k,b}(x) & \text{when } x \in I_{(l_k)} \setminus \{u\} \end{cases} \quad (5.19)$$

for each  $b$  and set

$$\tau_{c,d} = \phi_{c,d} \quad \text{all other } (c, d). \quad (5.20)$$

Cases  $k = 2$  and  $k = 3$  are explained in Appendix A. Then we can show  $\{\tau_{a,b}\} \in \mathcal{M}_A(\tilde{f}_i Y, s_i w)$  by using the fact  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ . Thus  $\tilde{f}_i Y$  satisfies the condition (A2) for  $s_i w$ .

**For  $\lambda$  of type (3).** By Lemma 4.16, it is enough to show that  $\tilde{f}_i Y$  satisfies condition (B2) for  $s_i w$ . Since  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , there exists  $\{\phi_{a,b}\} \in \mathcal{M}_B(Y, s_i w)$ . Fix a  $\{\phi_{a,b}\} \in \mathcal{M}_B(Y, s_i w)$ . If the family of maps  $\{\phi_{a,b}\} \in \mathcal{M}_B(\tilde{f}_i Y, s_i w)$ , then  $\tilde{f}_i Y$  satisfies the condition (B2) for  $s_i w$ . So we obtain  $\tilde{f}_i Y \in \mathcal{A}_{s_i w}(\lambda)$ . If, to the contrary, the family of maps  $\{\phi_{a,b}\} \notin \mathcal{M}_B(\tilde{f}_i Y, s_i w)$ , then  $\{\phi_{a,b}\}$  does not satisfy at least one of the conditions (B2-1) or (B2-2) for  $\tilde{f}_i Y$ .

**Case 1.** First, we consider the case when  $\{\phi_{a,b}\}$  satisfies the condition (B2-1) but does not satisfy the condition (B2-2) for  $\tilde{f}_i Y$ .

For this  $\lambda$ , we can easily check that this case does not come into being; i.e., any  $\{\phi_{a,b}\} \in \mathcal{M}_B(Y, s_i w)$  satisfies the condition (B2-2) for  $\tilde{f}_i Y$ . Thus  $\tilde{f}_i Y$  satisfies the condition (B2).

**Case 2.** Next we deal with the case when  $\{\phi_{a,b}\}$  does not satisfy condition (B2-1) for  $\tilde{f}_i Y$ . It means that we have the following cases.

One case is when  $1 \leq k \leq p' + 1$  and for some  $b$  with  $p' + 1 \leq b \leq p' + 2$  and  $k < b$ ,

$$c((\tilde{f}_i Y)_u^k) > c((E_{s_i w \lambda})_{\phi_{k,b}(u)}^b), \quad (5.21)$$

the other case is when  $k = p' + 2$  and for some  $a$  with  $1 \leq a \leq p' + 1$  and  $a < k$ ,

$$c((\tilde{f}_i Y)_u^k) > c((\tilde{f}_i Y)_v^a) \quad (5.22)$$

where  $v \in I_{(l_a)}$  satisfies  $\phi_{a,k}(v) = u$ .

For the case  $1 \leq k \leq p'$ , we have  $u = 1$  and  $c((E_{s_i w \lambda})_{\phi_{k,b}(1)}^b) = i - 1$ . And also we obtain  $\phi_{k,b}(1) + 1 \in I_{(l_b)}$  and  $c((E_{s_i w \lambda})_{\phi_{k,b}(1)+1}^b) = i$  from Remark 4.3. Set

$$\tau_{k,b}(1) = \phi_{k,b}(1) + 1 \quad (5.23)$$

for each  $b$  and set

$$\tau_{c,d} = \phi_{c,d} \quad \text{all other } (c, d). \quad (5.24)$$

Cases  $k = p' + 1$  and  $k = p' + 2$  are similar to the proof in Appendix A. Then we can show  $\{\tau_{a,b}\} \in \mathcal{M}_B(\tilde{f}_i Y, s_i w)$  by using the fact  $\{\phi_{a,b}\} \in \mathcal{M}_B(Y, s_i w)$ . Thus  $\tilde{f}_i Y$  satisfies the condition (B2).

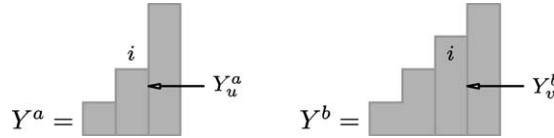
We have shown  $\tilde{f}_i Y$  satisfies (Y2) for  $s_i w$ . Thus  $\tilde{f}_i Y \in \mathcal{A}_{s_i w}(\lambda)$  as claimed and the induction on  $m$  is complete. We have shown that the right-hand side of (5.7) is contained in  $\mathcal{A}_{s_i w}(\lambda)$ .

**Step 2.** We shall now show the converse inclusion in (5.7).

Given  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , we need to show  $Y = \tilde{f}_i^m Y'$  for some  $Y' \in \mathcal{A}_w(\lambda)$  and  $m \geq 0$ . This is equivalent to showing  $\tilde{e}_i^m Y \in \mathcal{A}_w(\lambda)$  for some  $m \geq 0$ , given any  $Y \in \mathcal{A}_{s_i w}(\lambda)$ . For any  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , we shall show  $\tilde{e}_i^m Y \in \mathcal{A}_w(\lambda)$  by using induction on  $m$  where  $\varepsilon_i(Y) = m$ .

First, we shall show  $Y \in \mathcal{A}_w(\lambda)$ , given any  $Y \in \mathcal{A}_{s_i w}(\lambda)$  with  $\varepsilon_i(Y) = 0$ . Since  $\varepsilon_i(Y) = 0$ , the  $i$ -signature of  $Y$  has only  $+$ 's or is empty.

If  $Y$  does not have any removable  $i$ -block, then we have  $Y \preceq E_{w\lambda}$  by Remark 4.3. Next, if  $Y$  contains removable  $i$ -blocks, corresponding to each removable  $i$ -block, there must exist an  $i$ -admissible slot, which sits to the left of the removable  $i$ -block, so that the  $+$  and  $-$  signs placed under these columns cancel out, when obtaining the  $i$ -signature for  $Y$ . Let us suppose that  $a < b$ , the column  $Y_v^b$  is  $i$ -removable, and the column  $Y_u^a$  is  $i$ -admissible.



Since  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , there exist  $\phi_{a,b}$  satisfying condition (Y2-1) for  $s_i w$ . By (Y2-1), we know  $\phi_{a,b}(u) < v$  and  $c((E_{s_i w\lambda})_{\phi_{a,b}(u)}^b) \geq i - 1$ , so that we obtain  $c((E_{w\lambda})_{\phi_{a,b}(u)}^b) \geq i$ . Thus the removable  $i$ -block in  $Y_v^b$  is contained in  $E_{w\lambda}$ . Hence we have  $Y \preceq E_{w\lambda}$ .

Let us now show  $Y$  satisfies condition (Y2) for  $w$ . Since  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , there exist  $\{\phi_{a,b}\} \in \mathcal{M}(Y, s_i w)$ . Fix a family  $\{\phi_{a,b}\} \in \mathcal{M}(Y, s_i w)$ . We need only consider pairs  $(a, b)$  and  $u \in I_{(l_a)}$  such that  $c(Y_u^a) = c((E_{s_i w\lambda})_{\phi_{a,b}(u)}^b) = i$ . We may divide these into two cases; when  $u - 1 \in I_{(l_a)}$  and  $c(Y_{u-1}^a) = i - 1$ , or when  $Y_u^a$  is  $i$ -removable.

For the first case, we have  $c((E_{s_i w\lambda})_{\phi_{a,b}(u-1)}^b) = i - 1$ , so that  $(E_{s_i w\lambda})_{\phi_{a,b}(u)}^b$  is not  $i$ -removable and  $c((E_{w\lambda})_{\phi_{a,b}(u)}^b) = c((E_{s_i w\lambda})_{\phi_{a,b}(u)}^b)$ . Since  $\varepsilon_i(Y) = 0$ , for the second case, there exists a column  $Y_v^{a'}$  with  $a' < a$  and  $v \in I_{(l_{a'})}$  which is  $i$ -admissible. And since  $\{\phi_{a,b}\}$  satisfies condition (Y2-2),  $\phi_{a',b}(v) < \phi_{a,b}(u)$  or  $\phi_{a',b}(v) > \phi_{a,b}(u)$ . If  $\phi_{a',b}(v) < \phi_{a,b}(u)$ , then we have  $c((E_{s_i w\lambda})_{\phi_{a',b}(v)}^b) = i - 1$ , and then  $(E_{s_i w\lambda})_{\phi_{a,b}(u)}^b$  is not  $i$ -removable and  $c((E_{w\lambda})_{\phi_{a,b}(u)}^b) = c((E_{s_i w\lambda})_{\phi_{a,b}(u)}^b)$ . We shall deal with the case  $\phi_{a',b}(v) > \phi_{a,b}(u)$  in Appendix B.

We have shown that  $Y$  satisfies condition (Y2) for  $w$  and so that  $Y \in \mathcal{A}_w(\lambda)$ .

We now proceed with the induction step. Assume  $\tilde{e}_i^m Y \in \mathcal{A}_w(\lambda)$ , given any  $Y \in \mathcal{A}_{s_i w}(\lambda)$  with  $\varepsilon_i(Y) = m$ . Choose a  $Y \in \mathcal{A}_{s_i w}(\lambda)$  with  $\varepsilon_i(Y) = m + 1$ . Then  $\varepsilon_i(\tilde{e}_i Y) = m$ . If  $\tilde{e}_i Y \in \mathcal{A}_{s_i w}(\lambda)$ , then  $\tilde{e}_i^{m+1} Y \in \mathcal{A}_w(\lambda)$  by induction hypothesis, and the induction on  $m$  is complete. Thus we have only to show that  $\tilde{e}_i Y \in \mathcal{A}_{s_i w}(\lambda)$ .

Since  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , trivially we obtain  $\tilde{e}_i Y \preceq E_{s_i w\lambda}$ . We should now show that  $\tilde{e}_i Y$  satisfies the condition (Y2) for  $s_i w$ . This will be done for each type of  $\lambda$ , separately, using Lemma 4.16.

Let

$$\tilde{e}_i Y = [Y^1, \dots, Y^k \nearrow \boxed{i}, \dots, Y^p]. \quad (5.25)$$

We assume that the  $u$ th column of the  $k$ th component of  $\tilde{e}_i Y$  contains the  $i$ -block removed from  $Y^k$ .

$$(\tilde{e}_i Y)^k = Y^k \nearrow \boxed{i} = \text{diagram} \quad (\tilde{e}_i Y)_u^k$$

**For  $\lambda$  of type (2).** By Lemma 4.16, it is enough to show that  $\tilde{e}_i Y$  satisfies condition (A2) for  $s_i w$ . Since  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , there exists  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ . Fix a  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ . If the family of maps  $\{\phi_{a,b}\} \in \mathcal{M}_A(\tilde{e}_i Y, s_i w)$ , then  $\tilde{e}_i Y$  satisfies the condition (A2) for  $s_i w$ . So we obtain  $\tilde{e}_i Y \in \mathcal{A}_{s_i w}(\lambda)$ . If, to the contrary, the family of maps  $\{\phi_{a,b}\} \notin \mathcal{M}_A(\tilde{e}_i Y, s_i w)$ , then  $\{\phi_{a,b}\}$  does not satisfy at least one of the conditions (A2-1) or (A2-2) for  $\tilde{e}_i Y$ .

**Case 1.** First, we consider the case when  $\{\phi_{a,b}\}$  satisfies condition (A2-1) but does not satisfy the condition (A2-2) for  $\tilde{e}_i Y$ .

It means  $k = 1$  and for some  $v \in I_{(l_2)}$ ,

$$\phi_{2,3}(v) = \phi_{k,3}(u), \quad c((\tilde{e}_i Y)_v^2) > c((\tilde{e}_i Y)_u^k). \quad (5.26)$$

For this case,  $c((\tilde{e}_i Y)_v^2) = i$ . We also know that  $v - 1 \in I_{(l_2)}$  and  $c((\tilde{e}_i Y)_{v-1}^2) = i - 1$ , since  $\tilde{e}_i$  acts on  $Y^k$ . Set

$$\tau_{2,3}(x) = \begin{cases} \phi_{2,3}(v) - 1 & \text{when } x = v - 1, \\ \phi_{2,3}(x) & \text{when } x \in I_{(l_2)} \setminus \{v - 1\}. \end{cases} \quad (5.27)$$

For  $x \in I_{(l_k)}$ , set

$$\tau_{k,3}(x) = \begin{cases} \phi_{k,3}(x) - 1 & \text{when } \phi_{2,3}(v - 1) \leq \phi_{k,3}(x) \leq \phi_{k,3}(u), \\ \phi_{k,3}(x) & \text{otherwise.} \end{cases} \quad (5.28)$$

And finally set

$$\tau_{k,2} = \phi_{k,2}. \quad (5.29)$$

Note that since  $\{\phi_{a,b}\}$  satisfies the condition (A2-2) for  $Y$ ,

$$\{\phi_{k,3}(x) \mid \phi_{2,3}(v - 1) \leq \phi_{k,3}(x) \leq \phi_{k,3}(u), x \in I_{(l_k)}\} \cap \{\phi_{2,3}(v - 1)\} = \emptyset.$$

We can show  $\{\tau_{a,b}\} \in \mathcal{M}_A(\tilde{e}_i Y, s_i w)$  by using the fact  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ , so that  $\tilde{e}_i Y$  satisfies the condition (A2).

**Case 2.** Next we deal with the case when  $\{\phi_{a,b}\}$  does not satisfy condition (A2-1) for  $\tilde{e}_i Y$ .

It means that  $k = 1$  or 2, and for some  $b$  with  $k < b \leq 3$ ,

$$c((\tilde{e}_i Y)_u^k) < c((\tilde{e}_i Y)_{\phi_{k,b}(u)}^b). \quad (5.30)$$

For the case  $k = 2$ , we have  $c((\tilde{e}_i Y)_{\phi_{k,3}(u)}^3) = i$ . Since  $\tilde{e}_i$  act on  $Y^k$ ,  $\phi_{k,3}(u) - 1 \in I_{(l_3)}$  and  $c((\tilde{e}_i Y)_{\phi_{k,3}(u)-1}^3) = i - 1$ . Set

$$\tau_{k,3}(x) = \begin{cases} \phi_{k,3}(u) - 1 & \text{when } x = u, \\ \phi_{k,3}(x) & \text{when } x \in I_{(l_k)} \setminus \{u\}. \end{cases} \quad (5.31)$$

And set

$$\tau_{a,b} = \phi_{a,b} \quad \text{for all other } (a, b). \quad (5.32)$$

The case  $k = 1$  is explained in Appendix C. Then we can show  $\{\tau_{a,b}\} \in \mathcal{M}_A(\tilde{e}_i Y, s_i w)$  by using the fact that  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ . Thus  $\tilde{e}_i Y$  satisfies the condition (A2).

**For  $\lambda$  of type (3).** By Lemma 4.16, it is enough to show that  $\tilde{e}_i Y$  satisfies condition (B2) for  $s_i w$ . Since  $Y \in \mathcal{A}_{s_i w}(\lambda)$ , there exists  $\{\phi_{a,b}\} \in \mathcal{M}_B(Y, s_i w)$ . Fix a  $\{\phi_{a,b}\} \in \mathcal{M}_B(Y, s_i w)$ . If the family of maps  $\{\phi_{a,b}\} \in \mathcal{M}_B(\tilde{e}_i Y, s_i w)$ , then  $\tilde{e}_i Y$  satisfies the condition (B2). So we obtain  $\tilde{e}_i Y \in \mathcal{A}_{s_i w}(\lambda)$ . If, to the contrary, the family of maps  $\{\phi_{a,b}\} \notin \mathcal{M}_B(\tilde{e}_i Y, s_i w)$ , then  $\{\phi_{a,b}\}$  does not satisfy at least one of the conditions (B2-1) or (B2-2) for  $\tilde{e}_i Y$ .

**Case 1.** First, we consider the case when  $\{\phi_{a,b}\}$  satisfies condition (B2-1) but does not satisfy the condition (B2-2) for  $\tilde{e}_i Y$ .

It means that  $1 \leq k \leq p'$ ,  $b = p' + 1$ ,  $c = p' + 2$ , and for some  $v \in I_{(l_b)}$

$$\phi_{k,c}(u) = \phi_{b,c}(v), \quad c((\tilde{e}_i Y)_v^b) > c((\tilde{e}_i Y)_u^k). \quad (5.33)$$

Here,  $c((\tilde{e}_i Y)_v^b) = i$ . Since  $\tilde{e}_i$  acts on  $Y^k$ , we may divide these into two cases; when  $v - 1 \in I_{(l_b)}$  and  $c((\tilde{e}_i Y)_{v-1}^b) = i - 1$ , or when  $v - 1 \in I_{(l_b)}$ ,  $c((\tilde{e}_i Y)_{v-1}^b) < i - 1$ , and  $c((\tilde{e}_i Y)_1^{b'}) = i - 1$  for some  $b'$  with  $k < b' < b$ .

If  $c((\tilde{e}_i Y)_{v-1}^b) = i - 1$ , set

$$\tau_{k,c}(1) = \phi_{b,c}(v - 1), \quad (5.34)$$

and set

$$\tau_{a,b} = \phi_{a,b} \quad \text{for all other } (a, b). \quad (5.35)$$

On the other hand, if  $c((\tilde{e}_i Y)_{v-1}^b) < i - 1$  and  $c((\tilde{e}_i Y)_1^{b'}) = i - 1$  for some  $b'$  with  $k < b' < b$ , set

$$\tau_{k,c}(1) = \phi_{b',c}(1), \quad (5.36)$$

and set

$$\tau_{a,b} = \phi_{a,b} \quad \text{for all other } (a, b). \quad (5.37)$$

Then we can show  $\{\tau_{a,b}\} \in \mathcal{M}_B(\tilde{e}_i Y, s_i w)$  by using the fact  $\{\phi_{a,b}\} \in \mathcal{M}_B(Y, s_i w)$  so that  $\tilde{e}_i Y$  satisfies the condition (B2).

**Case 2.** Next we deal with the case when  $\{\phi_{a,b}\}$  does not satisfy condition (B2-1) for  $\tilde{e}_i Y$ .

It means that  $1 \leq k \leq p' + 1$  and for some  $b$  with  $p' + 1 \leq b \leq p' + 2$  and  $k < b$ ,

$$c((\tilde{e}_i Y)_u^k) < c((\tilde{e}_i Y)_{\phi_{k,b}(u)}^b). \quad (5.38)$$

Here,  $c((\tilde{e}_i Y)_{\phi_{k,b}(u)}^b) = i$ . Since  $\tilde{e}_i$  acts on  $Y^k$ , we may divide these into two cases; when  $\phi_{k,b}(u) - 1 \in I_{(l_b)}$  and  $c((\tilde{e}_i Y)_{\phi_{k,b}(u)-1}^b) = i - 1$ , or when  $\phi_{k,b}(u) - 1 \in I_{(l_b)}$ ,  $c((\tilde{e}_i Y)_{\phi_{k,b}(u)-1}^b) < i - 1$ , and  $c((\tilde{e}_i Y)_v^{b'}) = i - 1$  for some  $b'$  with  $k < b' < b$  and  $v \in I_{(l_{b'})}$ .

If  $c((\tilde{e}_i Y)_{\phi_{k,b}(u)-1}^b) = i - 1$ , set

$$\tau_{k,b}(x) = \begin{cases} \phi_{k,b}(u) - 1 & \text{when } x = u, \\ \phi_{k,b}(x) & \text{when } x \in I_{(l_k)} \setminus \{u\}, \end{cases} \quad (5.39)$$

for each  $b$ . And if  $c((\tilde{e}_i Y)_{\phi_{k,b}(u)-1}^b) < i - 1$  and  $c((\tilde{e}_i Y)_v^{b'}) = i - 1$  for some  $b'$  with  $k < b' < b$  and  $v \in I_{(l_{b'})}$ , set

$$\tau_{k,b}(1) = \phi_{b',b}(v) \quad (5.40)$$

for each  $b$ . Otherwise, set

$$\tau_{c,d} = \phi_{c,d} \quad \text{for all other } (c, d). \quad (5.41)$$

Then we can show  $\{\tau_{a,b}\} \in \mathcal{M}_B(\tilde{e}_i Y, s_i w)$  by using the fact  $\{\phi_{a,b}\} \in \mathcal{M}_B(Y, s_i w)$  so that  $\tilde{e}_i Y$  satisfies the condition (B2).

We have shown  $\tilde{e}_i Y$  satisfies (Y2) for  $s_i w$ . Thus  $\tilde{e}_i Y \in \mathcal{A}_{s_i w}(\lambda)$  as claimed and the induction on  $m$  is complete. We have shown that  $\mathcal{A}_{s_i w}(\lambda)$  is contained in the right-hand side of (5.7).

Through Steps 1 and 2, we have shown that (5.7) is true. This concludes the proof Theorem 4.11.

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## Appendix A

(1) In the case  $k = 2$ , we have  $c((E_{s_i w \lambda})_{\phi_{k,3}(u)}^3) = i - 1$ . And we can also obtain  $\phi_{k,3}(u) + 1 \in I_{(l_3)}$  and  $c((E_{s_i w \lambda})_{\phi_{k,3}(u)+1}^3) = i$  from Remark 4.3.

If for some  $v \in I_{(l_1)}$ ,  $\phi_{1,3}(v) = \phi_{k,3}(u) + 1$  and  $c(Y_v^1) < i$ , then set

$$\tau_{k,3}(x) = \begin{cases} \phi_{k,3}(u) + 1 & \text{when } x = u, \\ \phi_{k,3}(x) & \text{when } x \in I_{(l_k)} \setminus \{u\}, \end{cases} \quad (\text{A.1})$$

and

$$\tau_{1,3}(x) = \begin{cases} \phi_{k,3}(u) & \text{when } x = v, \\ \phi_{1,3}(x) & \text{when } x \in I_{(l_1)} \setminus \{v\}, \end{cases} \quad (\text{A.2})$$

and

$$\tau_{1,2} = \phi_{1,2}. \quad (\text{A.3})$$

For the other case, set

$$\tau_{k,3}(x) = \begin{cases} \phi_{k,3}(u) + 1 & \text{when } x = u, \\ \phi_{k,3}(x) & \text{when } x \in I_{(l_k)} \setminus \{u\} \end{cases} \quad (\text{A.4})$$

and

$$\tau_{c,d} = \phi_{c,d} \quad \text{for all other } (c, d). \quad (\text{A.5})$$

(2) Now we consider the  $k = 3$  case.

In this case, we have  $a = 1$  and for some  $v \in I_{(l_1)}$ ,  $\phi_{1,k}(v) = u$  and  $c((\tilde{f}_i Y)_v^1) = i - 1$ . And since  $\phi_{1,k}$  satisfies condition (A2-1) for  $Y$ , if  $v + 1 \in I_{(l_1)}$ , then  $c((\tilde{f}_i Y)_{v+1}^1) > i$ . Since  $\tilde{f}_i$  acts on  $Y^k$ , there exists a column  $Y_i^2$  which is  $i$ -removable.

Let  $\alpha$  be the number of  $x \in I_{(l_1)}$  which satisfies  $\phi_{2,k}(t) \leq \phi_{1,k}(x) \leq \phi_{1,k}(v)$ . We shall denote by  $x_1, \dots, x_\alpha$ , such  $x$ . Note that we may assume  $x_j = x_1 + (j - 1)$ . Similarly, we denote by  $x'_1, \dots, x'_\beta$ , the  $x \in I_{(l_2)}$  satisfying  $\phi_{2,k}(t) \leq \phi_{2,k}(x) \leq \phi_{1,k}(v)$ . We assume  $x'_j = x'_1 + (j - 1)$ . Here,  $\phi_{1,k}(v) = u$ , and  $\phi_{2,k}(t) < \phi_{1,k}(v)$  since  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ .

Now set

$$\tau_{1,k}(x) = \begin{cases} \phi_{2,k}(t) + j - 1 & \text{when } x = x_j \text{ for some } j = 1, \dots, \alpha, \\ \phi_{1,k}(x) & \text{otherwise.} \end{cases} \quad (\text{A.6})$$

And set

$$\tau_{2,k}(x) = \begin{cases} \phi_{1,k}(v) - \beta + j & \text{when } x = x'_j \text{ for some } j = 1, \dots, \beta, \\ \phi_{2,k}(x) & \text{otherwise.} \end{cases} \quad (\text{A.7})$$

Finally, set

$$\tau_{1,2} = \phi_{1,2}. \quad (\text{A.8})$$

Note that  $\{\phi_{2,k}(x) \mid x \in I_{(l_2)} \text{ with } \phi_{2,k}(t) \leq \phi_{2,k}(x) \leq \phi_{1,k}(v)\} \cap \{\phi_{1,k}(x) \mid x \in I_{(l_1)} \text{ with } \phi_{2,k}(t) \leq \phi_{1,k}(x) \leq \phi_{1,k}(v)\} = \emptyset$  since  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ .

## Appendix B

We consider the case  $\phi_{a',b}(v) > \phi_{a,b}(u)$ . First, for  $\lambda$  of type (2), by using Lemma 4.16 we only need to consider the case  $1 \leq a < b \leq 3$ ; i.e.,  $a' = 1$ ,  $a = 2$ , and  $b = 3$ .

Let  $\alpha$  be the number of  $x \in I_{(l_1)}$  which satisfies  $\phi_{a,b}(u) \leq \phi_{1,b}(x) \leq \phi_{1,b}(v)$ . We shall denote by  $x_1, \dots, x_\alpha$ , such  $x$ . Note that we may assume  $x_j = x_1 + (j - 1)$ . Similarly, we denote by  $x'_1, \dots, x'_\beta$ , the  $x \in I_{(l_a)}$  satisfying  $\phi_{a,b}(u) \leq \phi_{a,b}(x) \leq \phi_{1,b}(v)$ . We assume  $x'_j = x'_1 + (j - 1)$ . Now set

$$\tau_{1,b}(x) = \begin{cases} \phi_{a,b}(u) + j - 1 & \text{when } x = x_j \text{ for some } j = 1, \dots, \alpha, \\ \phi_{1,b}(x) & \text{otherwise.} \end{cases} \quad (\text{B.1})$$

And set

$$\tau_{a,b}(x) = \begin{cases} \phi_{1,b}(v) - \beta + j & \text{when } x = x'_j \text{ for some } j = 1, \dots, \beta, \\ \phi_{a,b}(x) & \text{otherwise.} \end{cases} \quad (\text{B.2})$$

Finally, set

$$\tau_{1,a} = \phi_{1,a}. \quad (\text{B.3})$$

Note that  $\{\phi_{a,b}(x) \mid x \in I_{(l_a)} \text{ with } \phi_{a,b}(u) \leq \phi_{a,b}(x) \leq \phi_{1,b}(v)\} \cap \{\phi_{1,b}(x) \mid x \in I_{(l_1)} \text{ with } \phi_{a,b}(u) \leq \phi_{1,b}(x) \leq \phi_{1,b}(v)\} = \emptyset$  since the family  $\{\phi_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$ . Then  $\{\tau_{a,b}\} \in \mathcal{M}_A(Y, s_i w)$  and  $\{\tau_{a,b}\} \in \mathcal{M}_A(Y, w)$ . Thus  $Y$  satisfies (A2) for  $w$ .

For  $\lambda$  of type (3), we may use a similar argument.

## Appendix C

We consider the  $k = 1$  case. Here,  $c((\tilde{e}_i Y)_{\phi_{k,b}(u)}^b) = i$ . For this case, we have  $\phi_{k,b}(u) - 1 \in I_{(l_b)}$  and  $c((\tilde{e}_i Y)_{\phi_{k,b}(u)-1}^b) = i - 1$ , or  $(\tilde{e}_i Y)_{\phi_{k,3}(u)}^3$  is  $i$ -removable and  $(\tilde{e}_i Y)_v^2$  is  $i$ -admissible for some  $v \in I_{(l_2)}$ .

If  $c((\tilde{e}_i Y)_{\phi_{k,b}(u)-1}^b) = i - 1$ , set

$$\tau_{k,b}(x) = \begin{cases} \phi_{k,b}(u) - 1 & \text{when } x = u, \\ \phi_{k,b}(x) & \text{when } x \in I_{(l_k)} \setminus \{u\} \end{cases} \quad (\text{C.1})$$

for each  $b$  and set

$$\tau_{c,d} = \phi_{c,d} \quad \text{for all other } (c, d). \quad (\text{C.2})$$

If  $(\tilde{e}_i Y)_{\phi_{k,3}(u)}^3$  is  $i$ -removable and  $(\tilde{e}_i Y)_v^2$  is  $i$ -admissible for some  $v \in I_{(l_2)}$ , then  $\phi_{2,3}(v) < \phi_{k,3}(u)$  since  $\{\phi_{a,b}\}$  satisfies (A2-1) for  $Y$ .

Let  $\alpha$  be the number of  $x \in I_{(l_k)}$  which satisfies  $\phi_{2,3}(v) \leq \phi_{k,3}(x) \leq \phi_{k,3}(u)$ . We shall denote by  $x_1, \dots, x_\alpha$ , such  $x$ . Note that we may assume  $x_j = x_1 + (j-1)$ . Similarly, we denote by  $x'_1, \dots, x'_\beta$ , the  $x \in I_{(l_2)}$  satisfying  $\phi_{2,3}(v) \leq \phi_{2,3}(x) \leq \phi_{k,3}(u)$ . We assume  $x'_j = x'_1 + (j-1)$ . Now set

$$\tau_{k,3}(x) = \begin{cases} \phi_{2,3}(v) + j - 1 & \text{when } x = x_j \text{ for some } j = 1, \dots, \alpha, \\ \phi_{k,3}(x) & \text{otherwise.} \end{cases} \quad (\text{C.3})$$

And set

$$\tau_{2,3}(x) = \begin{cases} \phi_{2,3}(v) + \alpha + j - 2 & \text{when } x = x'_j \text{ for some } j = 1, \dots, \beta, \\ \phi_{2,3}(x) & \text{otherwise.} \end{cases} \quad (\text{C.4})$$

Finally, set

$$\tau_{1,2} = \phi_{1,2}. \quad (\text{C.5})$$

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